Design of Engineering Experiments Part 5 – The 2^k Factorial Design

- Text reference, Chapter 6
- **Special case** of the general factorial design; *k* factors, all at two levels
- The two levels are usually called **low** and **high** (they could be either quantitative or qualitative)
- Very widely used in industrial experimentation
- Form a basic "building block" for other very useful experimental designs (DNA)
- Special (short-cut) methods for analysis
- We will make use of Design-Expert

The Simplest Case: The 2^2

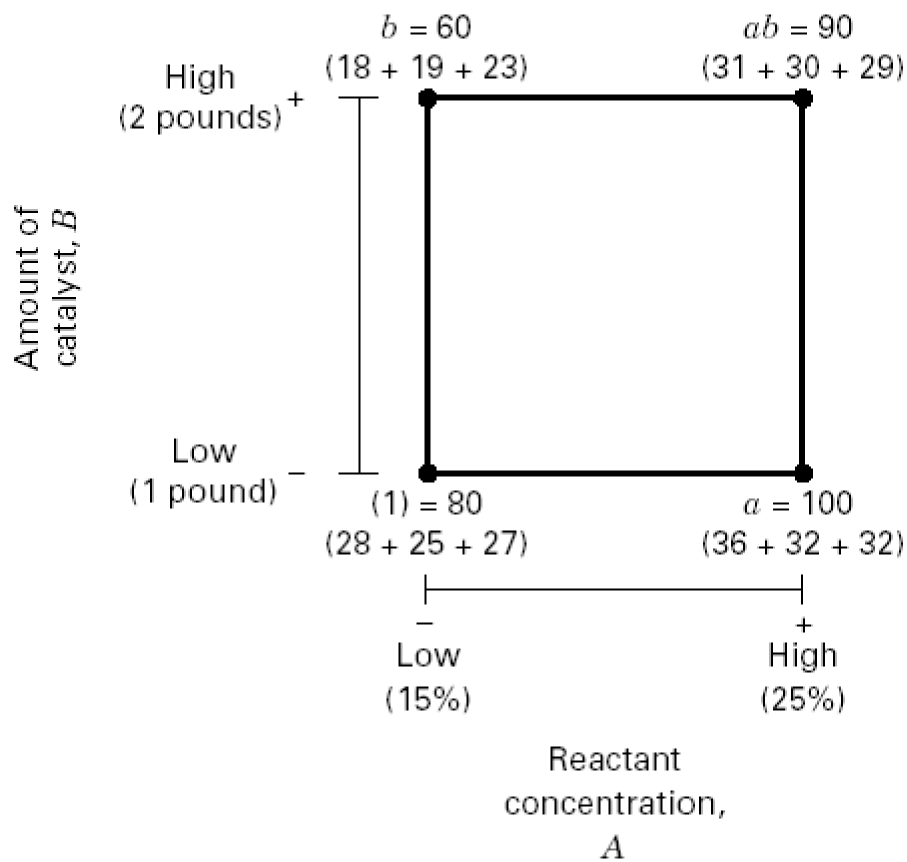


Figure 6-1 Treatment combinations in the 2^2 design.

"-" and "+" denote the low and high levels of a factor, respectively

Low and high are arbitrary terms

Geometrically, the four runs form the corners of a square

Factors can be quantitative or qualitative, although their treatment in the final model will be different

Chemical Process Example

Factor		Treatment	Replicate			
A	*B*	Combination	I	II	III	Total
−	−	*A* low, *B* low	28	25	27	80
+	−	*A* high, *B* low	36	32	32	100
−	+	*A* low, *B* high	18	19	23	60
+	+	*A* high, *B* high	31	30	29	90

A = reactant concentration, *B* = catalyst amount,
y = recovery

Analysis Procedure for a Factorial Design

- Estimate factor **effects**
- **Formulate** model
  - With replication, use full model
  - With an unreplicated design, use normal probability plots
- Statistical **testing** (ANOVA)
- **Refine** the model
- Analyze **residuals** (graphical)
- **Interpret** results

Estimation of Factor Effects

$$A = \bar{y}_{A^+} - \bar{y}_{A^-}$$
$$= \frac{ab+a}{2n} - \frac{b+(1)}{2n}$$
$$= \tfrac{1}{2n}[ab+a-b-(1)]$$
$$B = \bar{y}_{B^+} - \bar{y}_{B^-}$$
$$= \frac{ab+b}{2n} - \frac{a+(1)}{2n}$$
$$= \tfrac{1}{2n}[ab+b-a-(1)]$$
$$AB = \frac{ab+(1)}{2n} - \frac{a+b}{2n}$$
$$= \tfrac{1}{2n}[ab+(1)-a-b]$$

See textbook, pg. 205-206 For **manual** calculations

The effect estimates are: *A* = 8.33, *B* = -5.00, *AB* = 1.67

Practical interpretation?

Design-Expert analysis

Estimation of Factor Effects
Form Tentative Model

	Term	Effect	SumSqr	% Contribution
Model	Intercept			
Model	A	8.33333	208.333	64.4995
Model	B	-5	75	23.2198
Model	AB	1.66667	8.33333	2.57998
Error	Lack Of Fit	0	0	
Error	P Error	31.3333	9.70072	

Lenth's ME	6.15809
Lenth's SME	7.95671

Statistical Testing - ANOVA

Table 6-1 Analysis of Variance for the Experiment in Figure 6-1

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	P-Value
A	208.33	1	208.33	53.15	0.0001
B	75.00	1	75.00	19.13	0.0024
AB	8.33	1	8.33	2.13	0.1826
Error	31.34	8	3.92		
Total	323.00	11			

The *F*-test for the "model" source is testing the significance of the overall model; that is, is either *A*, *B*, or *AB* or some combination of these effects important?

Regression Model for the Process

	Coefficient		Standard	95% CI	95% CI	
Factor	Estimate	DF	Error	Low	High	VIF
Intercept	27.5	1	0.60604	26.12904	28.87096	
A-Concent	4.166667	1	0.60604	2.79571	5.537623	1
B-Catalyst	-2.5	1	0.60604	-3.87096	-1.12904	1
Final Equation in Terms of Coded Factors:						
	Conversior	=				
	27.5					
	4.166667	* A				
	-2.5	* B				
Final Equation in Terms of Actual Factors:						
	Conversior	=				
	18.33333					
	0.833333	* Concentration				
	-5	* Catalyst				

Residuals and Diagnostic Checking

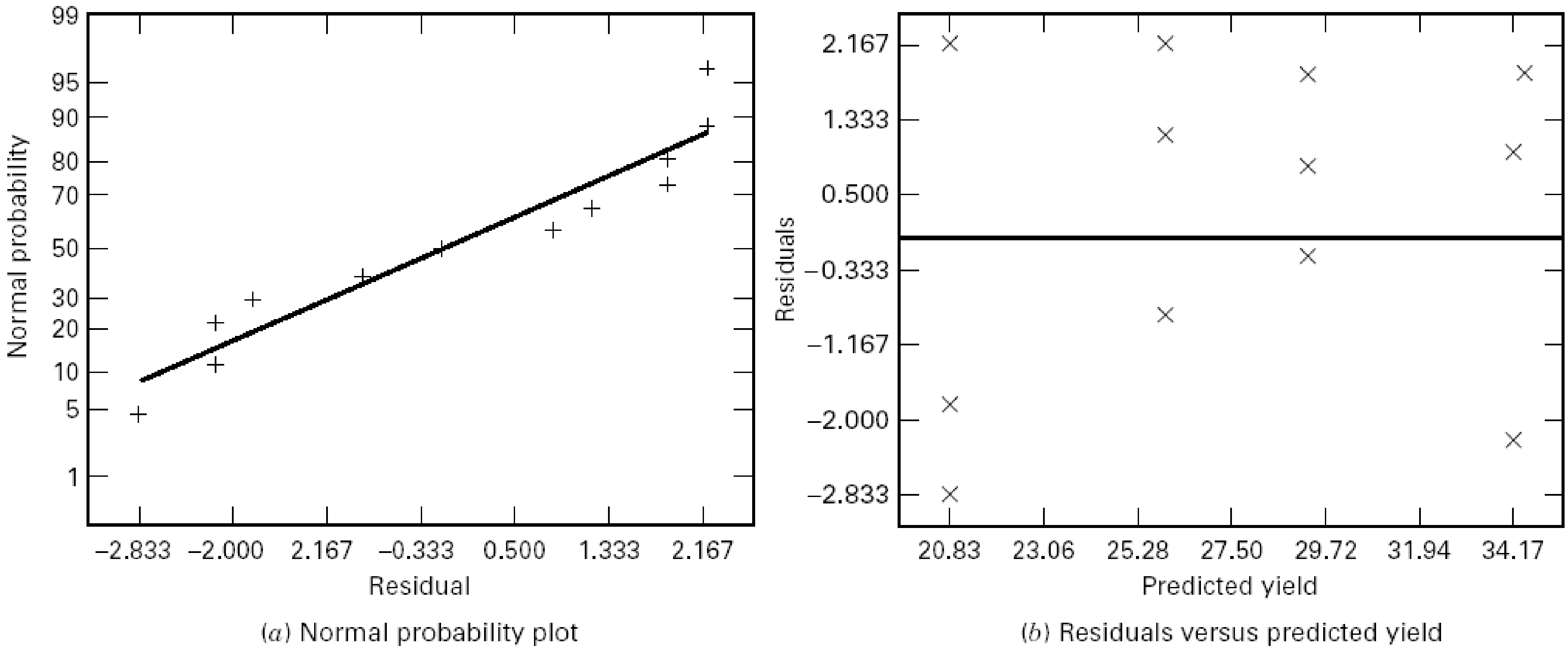


Figure 6-2 Residual plots for the chemical process experiment.

The Response Surface

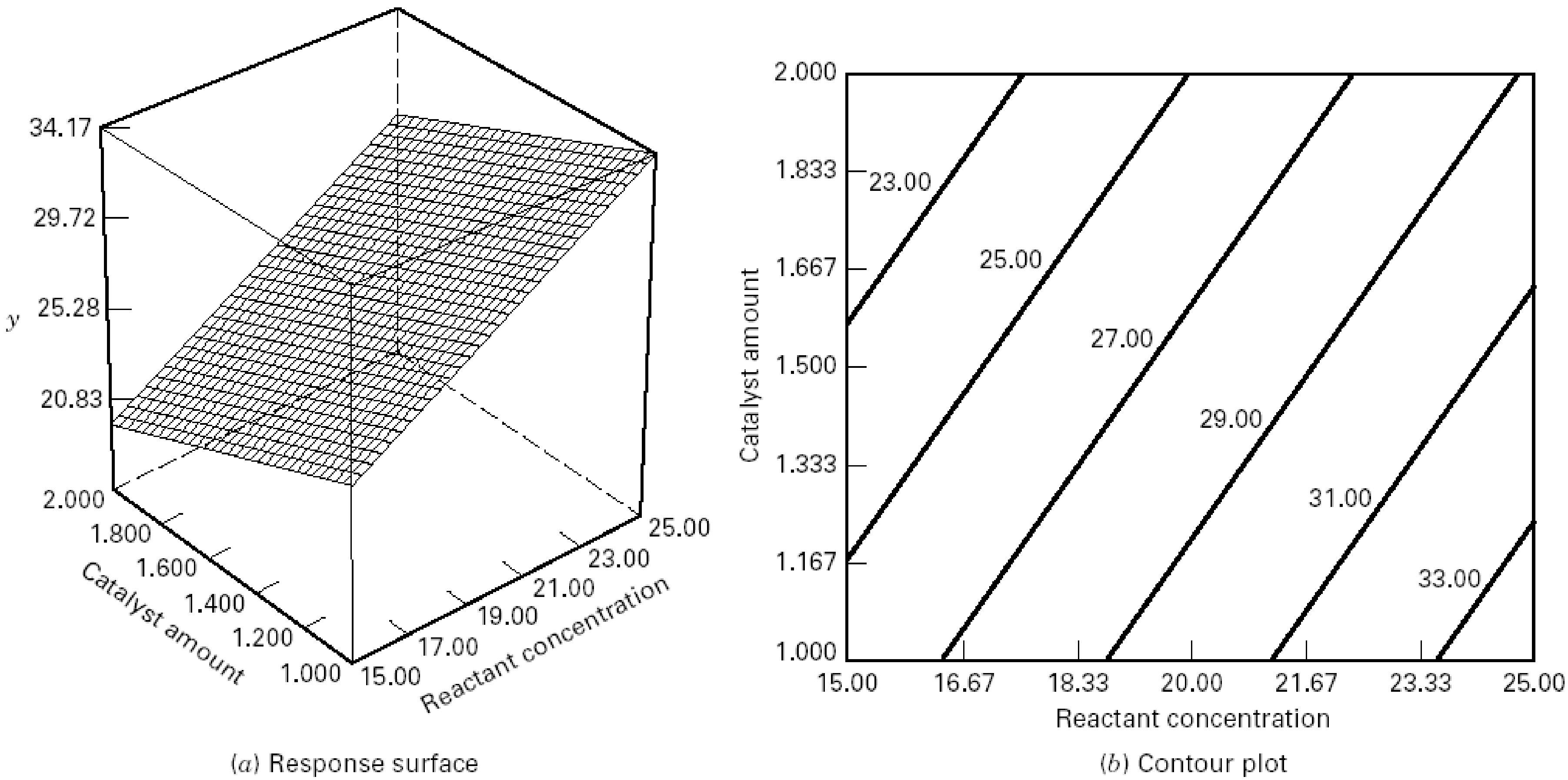
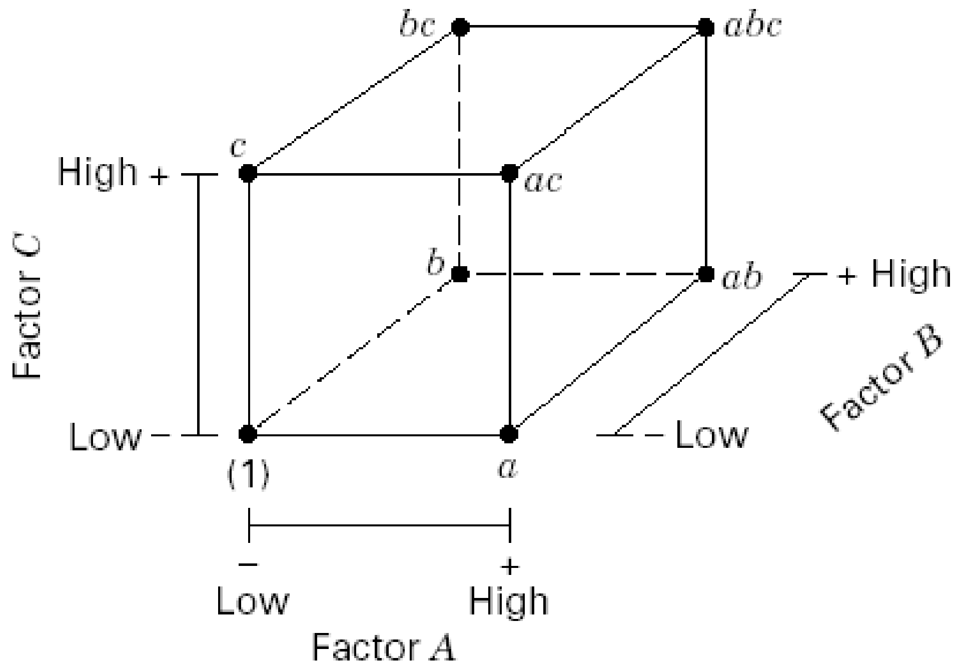


Figure 6-3 Response surface plot and contour plot of yield from the chemical process experiment.

The 2^3 Factorial Design



(a) Geometric view

Run	Factor A	B	C
1	−	−	−
2	+	−	−
3	−	+	−
4	+	+	−
5	−	−	+
6	+	−	+
7	−	+	+
8	+	+	+

(b) The design matrix

Figure 6-4 The 2^3 factorial design.

Effects in The 2^3 Factorial Design

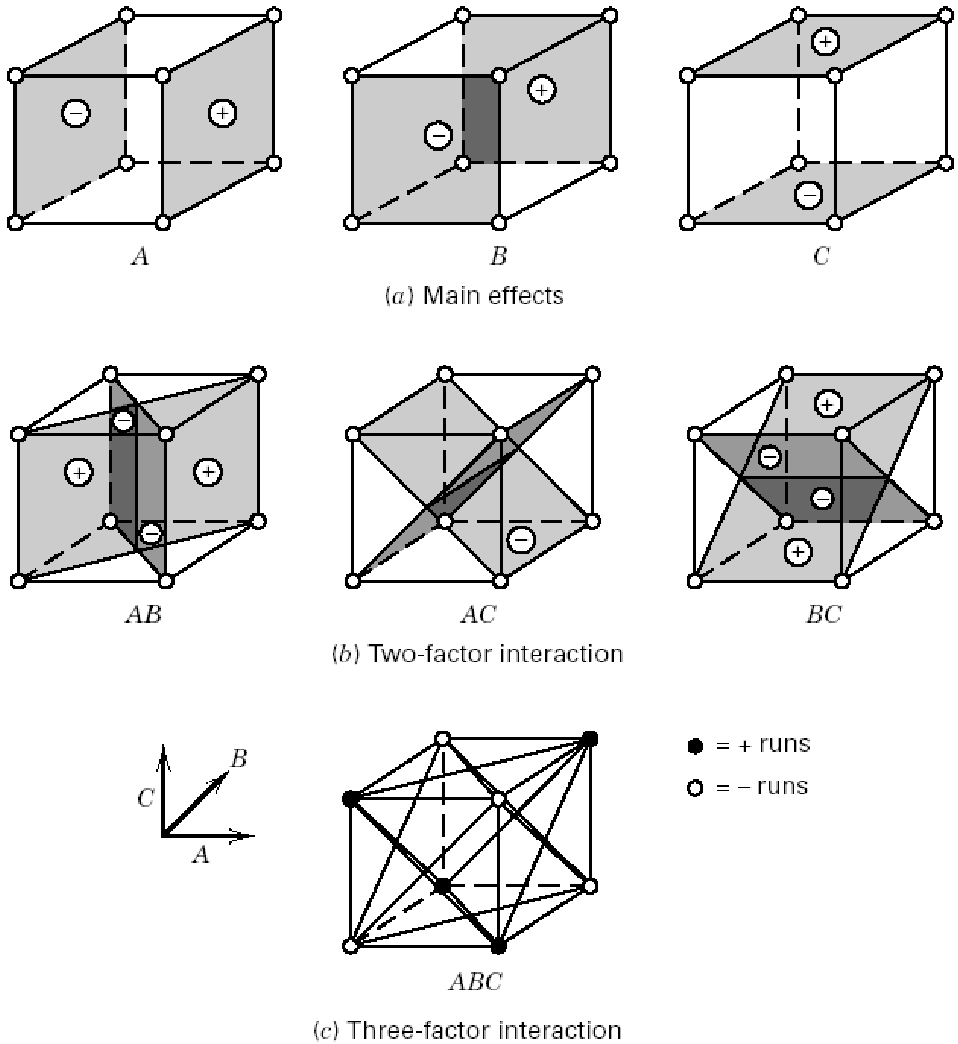


Figure 6-5 Geometric presentation of contrasts corresponding to the main effects and interactions in the 2^3 design.

$$A = \bar{y}_{A^+} - \bar{y}_{A^-}$$

$$B = \bar{y}_{B^+} - \bar{y}_{B^-}$$

$$C = \bar{y}_{C^+} - \bar{y}_{C^-}$$

etc, etc, ...

Analysis done via computer

An Example of a 2^3 Factorial Design

Table 6-4 The Plasma Etch Experiment, Example 6-1

	Coded Factors			Etch Rate			Factor Levels		
Run	*A*	*B*	*C*	Replicate 1	Replicate 2	Total		Low (−1)	High (+1)
1	−1	−1	−1	550	604	(1) = 1154	*A* (Gap, cm)	0.80	1.20
2	1	−1	−1	669	650	*a* = 1319	*B* (C_2F_6 flow, SCCM)	125	200
3	−1	1	−1	633	601	*b* = 1234	*C* (Power, W)	275	325
4	1	1	−1	642	635	*ab* = 1277			
5	−1	−1	1	1037	1052	*c* = 2089			
6	1	−1	1	749	868	*ac* = 1617			
7	−1	1	1	1075	1063	*bc* = 2178			
8	1	1	1	729	860	*abc* = 1589			

A = gap, *B* = Flow, *C* = Power, *y* = Etch Rate

Table of – and + Signs for the 2^3 Factorial Design (pg. 214)

Table 6-3 Algebraic Signs for Calculating Effects in the 2^3 Design

Treatment Combination	Factorial Effect							
	I	*A*	*B*	*AB*	*C*	*AC*	*BC*	*ABC*
(1)	+	−	−	+	−	+	+	−
a	+	+	−	−	−	−	+	+
b	+	−	+	−	−	+	−	+
ab	+	+	+	+	−	−	−	−
c	+	−	−	+	+	−	−	+
ac	+	+	−	−	+	+	−	−
bc	+	−	+	−	+	−	+	−
abc	+	+	+	+	+	+	+	+

Properties of the Table

- Except for column I, every column has an equal number of + and – signs
- The sum of the product of signs in any two columns is zero
- Multiplying any column by I leaves that column unchanged (identity element)
- The product of any two columns yields a column in the table:

$$A \times B = AB$$

$$AB \times BC = AB^2C = AC$$

- **Orthogonal** design
- Orthogonality is an important property shared by all factorial designs

Estimation of Factor Effects

Table 6-5 Effect Estimate Summary for Example 6-1

Factor	Effect Estimate	Sum of Squares	Percent Contribution
A	−101.625	41,310.5625	7.7736
B	7.375	217.5625	0.0409
C	306.125	374,850.0625	70.5373
AB	−24.875	2475.0625	0.4657
AC	−153.625	94,402.5625	17.7642
BC	−2.125	18.0625	0.0034
ABC	5.625	126.5625	0.0238

ANOVA Summary – Full Model

Table 6-6 Analysis of Variance for the Plasma Etching Experiment

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	*P*-Value
Gap (*A*)	41,310.5625	1	41,310.5625	18.34	0.0027
Gas flow (*B*)	217.5625	1	217.5625	0.10	0.7639
Power (*C*)	374,850.0625	1	374,850.0625	166.41	0.0001
AB	2475.0625	1	2475.0625	1.10	0.3252
AC	94,402.5625	1	94,402.5625	41.91	0.0002
BC	18.0625	1	18.0625	0.01	0.9308
ABC	126.5625	1	126.5625	0.06	0.8186
Error	18,020.5000	8	2252.5625		
Total	531,420.9375	15			

Model Coefficients – Full Model

Factor	Coefficient Estimated	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	776.06	1	11.87	748.70	803.42	
A-Gap	−50.81	1	11.87	−78.17	−23.45	1.00
B-Gas flow	3.69	1	11.87	−23.67	31.05	1.00
C-Power	153.06	1	11.87	125.70	180.42	1.00
AB	−12.44	1	11.87	−39.80	14.92	1.00
AC	−76.81	1	11.87	−104.17	−49.45	1.00
BC	−1.06	1	11.87	−28.42	26.30	1.00
ABC	2.81	1	11.87	−24.55	30.17	1.00

Refine Model – Remove Nonsignificant Factors

Table 6-7 (continued)

Response: Etch rate
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	*F* Value	Prob > *F*
Model	5.106E+005	3	1.702E+005	97.91	<0.0001
A	*41310.56*	*1*	*41310.56*	*23.77*	*0.0004*
C	*3.749E+005*	*1*	*3.749E+005*	*215.66*	*<0.0001*
AC	*94402.56*	*1*	*94402.56*	*54.31*	*<0.0001*
Residual	20857.75	12	1738.15		
Lack of Fit	*2837.25*	*4*	*709.31*	*0.31*	*0.8604*
Pure Error	*18020.50*	*8*	*2252.56*		
Cor Total	5.314E+005	15			

Std. Dev.	41.69	R-Squared	0.9608
Mean	776.06	Adj R-Squared	0.9509
C.V.	5.37	Pred R-Squared	0.9302
PRESS	37080.44	Adeq Precision	22.055

Model Coefficients – Reduced Model

Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	776.06	1	10.42	753.35	798.77	
A-Gap	−50.81	1	10.42	−73.52	28.10	1.00
C-Power	153.06	1	10.42	130.35	175.77	1.00
AC	−76.81	1	10.42	−99.52	−54.10	1.00

Model Summary Statistics for Reduced Model (pg. 222)

- R^2 and adjusted R^2

$$R^2 = \frac{SS_{Model}}{SS_T} = \frac{5.106\times10^5}{5.314\times10^5} = 0.9608$$

$$R^2_{Adj} = 1 - \frac{SS_E / df_E}{SS_T / df_T} = 1 - \frac{20857.75/12}{5.314\times10^5/15} = 0.9509$$

- R^2 for prediction (based on PRESS)

$$R^2_{\text{Pred}} = 1 - \frac{PRESS}{SS_T} = 1 - \frac{37080.44}{5.314\times10^5} = 0.9302$$

Model Summary Statistics (pg. 222)

- **Standard error** of model coefficients (full model)

$$se(\hat{\beta}) = \sqrt{V(\hat{\beta})} = \sqrt{\frac{\sigma^2}{n2^k}} = \sqrt{\frac{MS_E}{n2^k}} = \sqrt{\frac{2252.56}{2(8)}} = 11.87$$

- **Confidence interval** on model coefficients

$$\hat{\beta} - t_{\alpha/2, df_E} se(\hat{\beta}) \le \beta \le \hat{\beta} + t_{\alpha/2, df_E} se(\hat{\beta})$$

The Regression Model

Final Equation in Terms of Coded Factors:

Etch rate =
+776.06
−50.81 * A
+153.06 * C
−76.81 * A * C

Final Equation in Terms of Actual Factors:

Etch rate =
−5415.37500
+4354.68750 * Gap
+21.48500 * Power
−15.36250 * Gap * Power

Model Interpretation

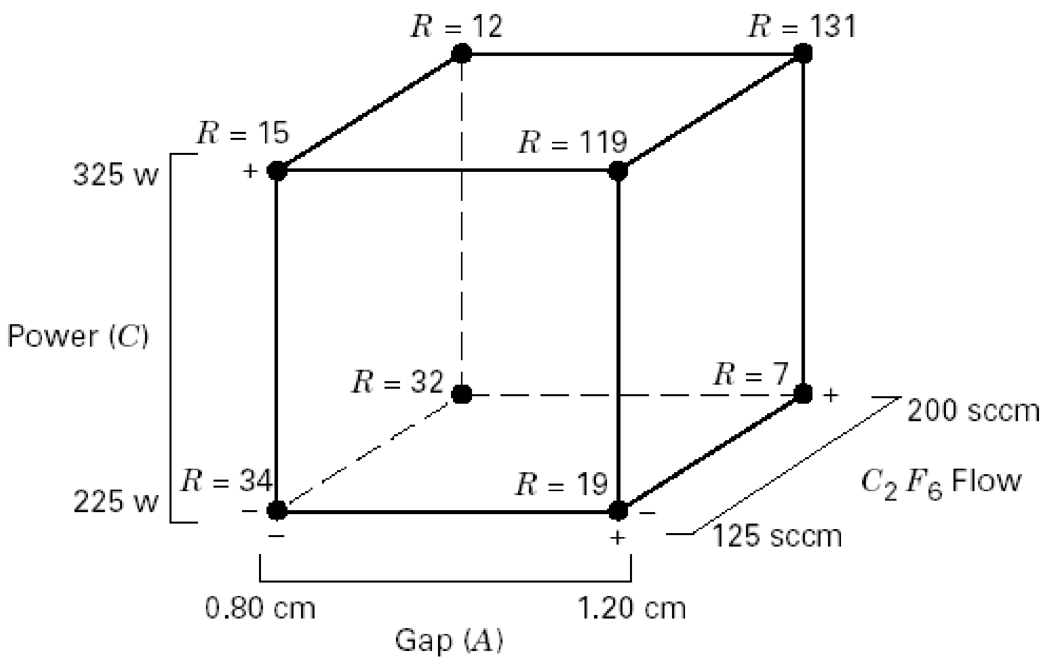


Figure 6-8 Ranges of etch rates for Example 6-1.

Cube plots are often useful visual displays of experimental results

The General 2^k Factorial Design

- Section 6-4, pg. 224, Table 6-9, pg. 225
- There will be k main effects, and

$$\binom{k}{2} \text{ two-factor interactions}$$

$$\binom{k}{3} \text{ three-factor interactions}$$

$$\vdots$$

$$1\ k-\text{factor interaction}$$