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Multi-Criteria Decision Making: An Operations Research Approach

E. Triantaphyllou, B. Shu, S. Nieto Sanchez, and T. Ray

Department of Industrial and Manufacturing Systems Engineering
3128 CEBA Building
Louisiana State University
Baton Rouge, LA 70803-6409, U.S.A.

E-mail: trianta@lsu.edu

Abstract: The core of operations research is the development of approaches for optimal decision making. A prominent class of such problems is multi-criteria decision making (MCDM). The typical MCDM problem deals with the evaluation of a set of alternatives in terms of a set of decision criteria. This paper provides a comprehensive survey of some methods for eliciting data for MCDM problems and also for processing such data.

Key words: Decision making, optimization, pairwise comparisons, sensitivity analysis, operations research.

1 Multi-Attribute Decision Making: A General Overview

Multi-Attribute Decision Making is the most well known branch of decision making. It is a branch of a general class of Operations Research (or OR) models which deal with decision problems under the presence of a number of decision criteria. This super class of models is very often called multi-criteria decision making (or MCDM). According to many authors (see, for instance, [Zimmermann, 1991]) MCDM is divided into Multi-Objective Decision Making (or MODM) and Multi-Attribute Decision Making (or MADM).

MODM studies decision problems in which the decision space is continuous. A typical example is mathematical programming problems with multiple objective functions. The first reference to this problem, also known as the *"vector-maximum"* problem, is attributed to [Kuhn and Tucker, 1951]. On the other hand, MADM concentrates on problems with discrete decision spaces. In these problems the set of decision alternatives has been predetermined.

Although MADM methods may be widely diverse, many of them have certain aspects in common [Chen and Hwang, 1992]. These are the notions of alternatives, and attributes (or criteria, goals) as described next.

Alternatives:
Alternatives represent the different choices of action available to the decision maker. Usually, the set of alternatives is assumed to be finite, ranging from several to hundreds. They are supposed to be screened, prioritized and eventually ranked.

Multiple attributes:

Each MADM problem is associated with multiple attributes. Attributes are also referred to as *"goals"* or *"decision criteria"*. Attributes represent the different dimensions from which the alternatives can be viewed.

In cases in which the number of attributes is large (e.g., more than a few dozens), attributes may be arranged in a **hierarchical** manner. That is, some attributes may be major attributes. Each major attribute may be associated with several sub-attributes. Similarly, each sub-attribute may be associated with several sub-sub-attributes and so on. Although some MADM methods may explicitly consider a hierarchical structure in the attributes of a problem, most of them assume a single level of attributes (e.g., no hierarchical structure).

Conflict among attributes:
Since different attributes represent different dimensions of the alternatives, they may conflict with each other. For instance cost may conflict with profit, etc.

Incommensurable units:
Different attributes may be associated with different units of measure. For instance, in the case of buying a used car, the attributes *"cost"* and *"mileage"* may be measured in terms of dollars and thousands of miles, respectively. It is this nature of having to consider different units which makes MADM to be intrinsically hard to solve.

Decision weights:
Most of the MADM methods require that the attributes be assigned weights of importance. Usually, these weights are normalized to add up to one. How these weights can be determined is described in section 6.2.

Decision matrix:
An MADM problem can be easily expressed in matrix format. A decision matrix **A** is an ($M \times N$) matrix in which element a_{ij} indicates the performance of alternative A_i when it is evaluated in terms of decision criterion C_j, (for i = 1,2,3,..., M, and j = 1,2,3,..., N). It is also assumed that the decision maker has determined the weights of relative performance of the decision criteria (denoted as W_j, for j = 1,2,3,..., N). This information is best summarized in figure 1. Given the previous definitions, then the general MADM problem can be defined as follows [Zimmermann, 1991]:

Definition 1-1:
Let $A = \{ A_i,$ for $i = 1,2,3,... ,M\}$ be a (finite) set of decision alternatives and $G = \{g_i,$ for $j = 1,2,3,..., N\}$ a (finite) set of goals according to which the desirability of an action is judged. Determine the optimal alternative A^* with the highest degree of desirability with respect to all relevant goals g_i.

	Criteria				
	C_1	C_2	C_3	...	C_N
Alt.	W_1	W_2	W_3	...	W_N
A_1	a_{11}	a_{12}	a_{13}	...	a_{1N}
A_2	a_{21}	a_{22}	a_{23}	...	a_{2N}
A_3	a_{31}	a_{32}	a_{33}	...	a_{3N}
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
A_M	a_{M1}	a_{M2}	a_{M3}	...	a_{MN}

Figure 1: A Typical Decision Matrix.

Very often, however, in the literature the goals g_i are also called decision criteria, or just criteria (since the alternatives need to be judged (evaluated) in terms of these goals). Another equivalent term is attributes. Therefore, the terms MADM and MCDM have been used very often to mean the same class of models (i.e., MADM). For these reasons, in this paper we will use the terms MADM and MCDM to denote the same concept.

2 Classification of MCDM Methods

As it was stated in the previous section, there are many MADM methods available in the literature. Each method has its own characteristics. There are many ways one can classify MADM methods. One way is to classify them according to the type of the data they use. That is, we have **deterministic**, **stochastic**, or **fuzzy** MADM methods (for an overview of fuzzy MADM methods see [Chen and Hwang, 1992]). However, there may be situations which involve combinations of all the above (such as stochastic and fuzzy data) data types.

Another way of classifying MADM methods is according to the number of decision makers involved in the decision process. Hence, we have **single** decision maker MADM methods and **group** decision making MADM (for more information on the later class, the interested reader may want to check the journal of **Group Decision Making**). In this paper we concentrate our attention on single decision maker deterministic MADM methods.

In [Chen and Hwang, 1992] deterministic -- single decision maker -- MADM methods were also classified according to the type of information and the salient features of the information. The WSM, AHP, revised AHP, WPM, and TOPSIS methods are the ones which are used mostly in practice today and are described in later sections. Finally, it should be stated here that there are many other alternative ways for classifying MADM methods [Chen and Hwang, 1992]. However, the previous ones are the most widely used approaches in the MADM literature.

3 Some MCDM Application Areas

Some of the industrial engineering applications of MCDM include the use of decision analysis in integrated manufacturing [Putrus, 1990], in the evaluation of technology investment decisions [Boucher and McStravic, 1991], in flexible manufacturing systems [Wabalickis, 1988], layout design [Cambron and Evans, 1991], and also in other engineering problems [Wang and Raz, 1991]. As an illustrative application consider the case in which one wishes to upgrade the computer system of a computer integrated manufacturing (CIM) facility. There is a number of different configurations available to choose from. The different systems are the alternatives. A decision should also consider issues such as: cost, performance characteristics (i.e., CPU speed, memory capacity, RAM size, etc.), availability of software, maintenance, expendability, etc. These may be some of the decision criteria for this problem. In the above problem we are interested in determining the **best alternative** (i.e., computer system). In some other situations, however, one may be interested in determining the relative importance of all the alternatives under consideration. For instance, if one is interested in funding a set of competing projects (which now are the alternatives), then the relative importance of these projects is required (so the budget can be distributed proportionally to their relative importances).

Multi-criteria decision-making (MCDM) plays a critical role in many real life problems. It is not an exaggeration to argue that almost any local or federal government, industry, or business activity involves, in one way or the other, the evaluation of a set of alternatives in terms of a set of decision criteria. Very often these criteria are conflicting with each other. Even more often the pertinent data are very expensive to collect.

4 Multi-Criteria Decision Making Methods

4.1 Background Information

With the continuing proliferation of decision methods and their modifications, it is important to have an understanding of their comparative value. Each of the methods uses numeric techniques to help decision makers choose among a discrete set of alternative decisions. This is achieved on the basis of the impact of the alternatives on certain criteria and thereby on the overall utility of the decision maker(s).

Despite the criticism that multi-dimensional methods have received, some of them are widely used. The weighted sum model (or WSM) is the earliest and probably the most widely used method. The weighted product model (or WPM) can be considered as a modification of the WSM, and has been proposed in order to overcome some of its weaknesses. The analytic hierarchy process (or AHP), as proposed by Saaty [Saaty, 1980, 1983, 1990, and

1994], is a later development and it has recently become increasingly popular. Professors Belton and Gear [1983] suggested a modification to the AHP that appears to be more powerful than the original approach. Some other widely used methods are the ELECTRE [Benayoun, *et al.*, 1966] and TOPSIS [Hwang and Yoon, 1981]. In the sub-section that follows these methods are presented in detail.

4.2 Description of Some MCDM Methods

There are three steps in utilizing any decision-making technique involving numerical analysis of alternatives:

1) *Determining the relevant criteria and alternatives.*
2) *Attaching numerical measures to the relative importance of the criteria and to the impacts of the alternatives on these criteria.*
3) *Processing the numerical values to determine a ranking of each alternative.*

This section is **only** concerned with the effectiveness of the four methods in performing step 3. The central decision problem examined in this paper is described as follows. Given is a set of M alternatives: $A_1, A_2, A_3, ..., A_M$ and a set of N decision criteria $C_1, C_2, C_3, ..., C_N$ and the data of a decision matrix as the one described in Figure 1. Then the problem is to rank the alternatives in terms of their total preferences when all the decision criteria are considered simultaneously.

4.2.1 The Weighted Sum Model

The weighted sum model (or WSM) is probably the most commonly used approach, especially in single dimensional problems. If there are M alternatives and N criteria then, the best alternative is the one that satisfies (in the maximization case) the following expression [Fishburn, 1967]:

$$A^*_{WSM} = \max_i \sum_{j=1}^{N} q_{ij} w_j, \quad for \quad i = 1,\ 2,\ 3,\ ...,\ M. \tag{4-1}$$

where: A_{WSM}* is the WSM score of the best alternative, N is the number of decision criteria, a_{ij} is the actual value of the *i-th* alternative in terms of the *j-th* criterion, and W_j is the weight of importance of the *j*-th criterion.

The assumption that governs this model is the **additive utility assumption.** That is, the total value of each alternative is equal to the sum of products given as (4-1). In single-dimensional cases, in which all the units are the same (e.g., dollars, feet, seconds), the WSM can be used without difficulty. Difficulty with this method emerges when it is applied to multi-dimensional decision-making problems. Then, in combining different dimensions, and consequently different units, the additive utility assumption is violated and the result is equivalent to *"adding apples and oranges"*.

Example 4-1:

Suppose that an MCDM problem involves four criteria, *which are expressed in exactly the same unit*, and three alternatives. The relative weights of the four criteria were determined to be: W_1 = 0.20, W_2 = 0.15, W_3 = 0.40, and W_4 = 0.25. The corresponding a_{ij} values are assumed to be as follows:

$$A = \begin{bmatrix} 25 & 20 & 15 & 30 \\ 10 & 30 & 20 & 30 \\ 30 & 10 & 30 & 10 \end{bmatrix}.$$

Therefore, the data (i.e., decision matrix) for this MCDM problem are as follows:

Criteria

C_1 C_2 C_3 C_4

Alt. (	0.20	0.15	0.40	0.25)
A_1	25	20	15	30
A_2	10	30	20	30
A_3	30	10	30	10

When formula (4-1) is applied on the previous data, the scores of the three alternatives are:

A_1(WSM score) = 25×0.20 + 20×0.15 + 15×0.40 + 30×0.25 = 21.50.

Similarly, A_2(WSM score) = 22.00,

and A_3(WSM score) = 20.00.

Therefore, the best alternative (in the maximization case) is alternative A_2 (because it has the highest WSM score; 22.00). Moreover, the following ranking is derived: $A_2 > A_1 > A_3$ (where ">" stands for *"better than"*). ■

4.2.2 The Weighted Product Model

The weighted product model (or WPM) is very similar to the WSM. The main difference is that instead of addition in the model there is multiplication. Each alternative is compared with the others by multiplying a number of ratios, one for each criterion. Each ratio is raised to the power equivalent to the relative weight of the corresponding criterion. In general, in order to compare the alternatives A_K and A_L, the following product (Bridgman [1922] and Miller and Starr [1969]) has to be calculated:

$$R(A_K/A_L) \quad = \quad \prod_{j=1}^{N} (a_{Kj}/a_{Lj})^{w_j}, \tag{4-2}$$

where: N is the number of criteria, a_{ij} is the actual value of the *i-th* alternative in terms of the *j-th* criterion, and W_j is the weight of importance of the *j-th* criterion.

If the term $R(A_K / A_L)$ is greater than to one, then alternative A_K is more desirable than alternative A_L (in the maximization case). The best alternative is the one that is better than or at least equal to all the other alternatives.

The WPM is sometimes called **dimensionless analysis** because its structure eliminates any units of measure. Thus, the WPM can be used in single- and multi-dimensional decision-making problems. An advantage of the method is that instead of the actual values it can use relative ones. This is true because:

$$\frac{a_{Kj}}{a_{Lj}} \quad = \quad \frac{a_{Kj}/\sum_{i=1}^{N} a_{Ki}}{a_{Lj}/\sum_{i=1}^{N} a_{Li}} \quad = \quad \frac{a'_{Kj}}{a'_{Lj}}. \tag{4-3}$$

A relative value a'_{Kj} is calculated by using the formula: $a'_{Kj} = a_{Kj}/\sum_{i=1}^{N} a_{Ki}$ where the a_{Kj}'s are the actual values.

Example 4-2:

Consider the problem presented in the previous example 4-1 (note that now the restriction to express all criteria in terms of the same unit is not needed). When the WPM is applied, then the following values are derived:

$R(A_1/A_2) = (25/10)^{0.20} \times (20/30)^{0.15} \times (15/20)^{0.40} \times (30/30)^{0.25} = 1.007 > 1.$

Similarly, $R(A_1/A_3) = 1.067 > 1,$

and $R(A_2/A_3) = 1.059 > 1.$

Therefore, the best alternative is A_1, since it is superior to all the other alternatives. Moreover, the ranking of these alternatives is as follows: $A_1 > A_2 > A_3$. ■

An alternative approach is one to use only products without ratios. That is, to use the following variant of formula (4-2):

$$P(A_K) = \prod_{j=1}^{N} (a_{Kj})^{w_j}, \qquad (4\text{-}4)$$

Then, when the previous data are used, exactly the same ranking is derived.

4.2.3 The Analytic Hierarchy Process

The analytic hierarchy process (or AHP) ([Saaty, 1980, 1983, 1990, and 1994]) is based on decomposing a complex MCDM problem into a system of hierarchies (more on these hierarchies can be found in [Saaty, 1980]). The final step in the AHP deals with the structure of an $M \times N$ matrix (where M is the number of alternatives and N is the number of criteria). This matrix is constructed by using the relative importances of the alternatives in terms of each criterion. The vector (a_{i1}, a_{i2}, a_{i3}, ..., a_{iN}) for each i is the principal eigenvector of an $N \times N$ reciprocal matrix which is determined by pairwise comparisons of the impact of the M alternatives on the *i-th* criterion (more on this, and some other related techniques, is presented in section 6).

Some evidence is presented in [Saaty, 1980] which supports the technique for eliciting numerical evaluations of qualitative phenomena from experts and decision makers. However, we are not concerned here with the possible advantages and disadvantages of the use of pairwise comparisons and the eigenvector method for determining values for the a_{ij}'s. Instead, we examine the method used in AHP to process the a_{ij} values **after** they have been determined. The entry a_{ij}, in the $M \times N$ matrix, represents the relative value of the alternative A_i when it is considered in terms of criterion C_j. In the original AHP the sum $\sum_{i=1}^{N} a_{ij}$ is equal to one.

According to AHP the best alternative (in the maximization case) is indicated by the following relationship (4-5):

$$A^*_{AHP} = \max_i \sum_{j=1}^{N} q_{ij} w_j, \quad for \quad i = 1,\ 2,\ 3,\ ...,\ M. \qquad (4\text{-}5)$$

The similarity between the WSM and the AHP is evident. The AHP uses relative values instead of actual ones. Thus, it can be used in single- or multi-dimensional decision making problems.

Example 4-3:

Again, consider the data used in the previous two examples (note that as in the WPM case the restriction to express all criteria in terms of the same unit is not needed). The AHP uses a series of pairwise comparisons (more on this can be found in section 6) to determine the relative performance of each alternative in terms of each one of the decision criteria. In other words, instead of the absolute data, the AHP would use the following relative data:

	Criteria			
	C_1	C_2	C_3	C_4
Alt. (	0.20	0.15	0.40	0.25)
A_1	25/65	20/55	15/65	30/65
A_2	10/65	30/55	20/65	30/65
A_3	30/65	5/55	30/65	5/65

That is, the columns in the decision matrix have been normalized to add up to 1. When formula (4-5) is applied on the previous data, the following scores are derived:

A_1(AHP score) = (25/65)×0.20 + (20/55)×0.15 + (15/65)×0.40 + (30/65)×0.25 = 0.34.

Similarly, A_2(AHP score) = 0.35,
and A_3(AHP score) = 0.31.

Therefore, the best alternative (in the maximization case) is alternative A_2 (because it has the highest AHP score; 0.35). Moreover, the following ranking is derived: $A_2 > A_1 > A_3$. ■

4.2.4 The Revised Analytic Hierarchy Process

Belton and Gear [1983] proposed a revised version of the AHP model. They demonstrated that an inconsistency can occur when the AHP is used. They presented a numerical example which deals with three criteria and three alternatives. In that example the indication of the best alternative changes when an identical alternative to one of the nonoptimal alternatives is introduced now creating four alternatives. According to the authors the root for that inconsistency is the fact that the relative values for each criterion sum up to one. Instead of having the relative values of the alternatives $A_1, A_2, A_3, ..., A_M$ sum up to one, they propose to divide each relative value by the maximum value of the relative values. In particular, they elaborated on the following example.

Example 4-4 *(from [Belton and Gear, 1983], p. 228):*
Suppose that the actual data of an MCDM problem with three alternatives and three criteria are as follows:

	Criteria		
	C_1	C_2	C_3
Alt. (	1/3	1/3	1/3)
A_1	1	9	8
A_2	9	1	9
A_3	1	1	1

Observe that in real life problems the decision maker may never know the previous real data. Instead, he/she can use the method of pairwise comparisons (as described in section 6) to derive the relative data. When the AHP is applied on the previous data, the following decision matrix with the relative data is derived:

	Criteria		
	C_1	C_2	C_3
Alt. (	1/3	1/3	1/3)
A_1	1/11	9/11	8/18
A_2	9/11	1/11	9/18
A_3	1/11	1/11	1/18

Therefore, it can be easily verified that the vector with the final AHP scores, is: (0.45, 0.47, 0.08). That is, the three alternatives are ranked as follows: $A_2 > A_1 > A_3$.

Next, we introduce a new alternative, say A_4, which is an **identical copy** of the existing alternative A_2 (i.e., $A_2 . A_4$). Furthermore, it is also assumed that the relative weights of importance of the three criteria remain the same (i.e., 1/3, 1/3, 1/3). When the new alternative A_4 is considered, it can be easily verified that the new decision matrix is as follows:

	Criteria		
	C_1	C_2	C_3
Alt. (	1/3	1/3	1/3)
A_1	1/20	9/12	8/27
A_2	9/20	1/12	9/27
A_3	1/20	1/12	1/27
A_4	9/20	1/12	9/27

Similarly as above, it can be verified that the vector with the final AHP scores, is: (0.37, 0.29, 0.06, 0.29). That is, the four alternatives are ranked as follows: $A_1 > A_2 . A_4 > A_3$. The authors claim that this result is in logical contradiction with the previous result (in which $A_2 > A_1$).

When the **revised** AHP is applied on the last data, the following decision matrix is derived:

	Criteria		
	C_1	C_2	C_3
Alt. (	1/3	1/3	1/3)
A_1	1/9	1	8/9
A_2	1	1/9	1
A_3	1/9	1/9	1/9
A_4	1	1/9	1

The vector with the final scores, is: (2/3, 19/27, 1/9, 19/27). That is, the four alternatives are ranked as follows: $A_2 . A_4 > A_1 > A_3$. The last ranking is, obviously, the desired one. ■

The revised AHP was sharply criticized by Saaty [1990]. He claimed that identical alternatives should not be considered in the decision process. However [Triantaphyllou and Mann, 1989] have demonstrated that similar logical contradictions are possible with the original AHP, as well as with the revised AHP, when non-identical alternatives are introduced.

4.2.5 The ELECTRE Method

The ELECTRE (for *Elimination and Choice Translating Reality*; English translation from the French original) method was first introduced in [Benayoun, *et al.*, 1966]. The basic concept of the ELECTRE method is to deal with *"outranking relations"* by using pairwise comparisons among alternatives under each one of the criteria separately. The outranking relationship of $A_i \, {}_{v} \, A_j$ describes that even when the *i*-th alternative does not dominate the *j*-th alternative quantitatively, then the decision maker may still take the risk of regarding A_i as almost surely better than A_j [Roy, 1973]. Alternatives are said to be dominated, if there is another alternative which excels them in one or more attributes and equals in the remaining attributes.

The ELECTRE method begins with pairwise comparisons of alternatives under each criterion. Using physical or monetary values $g_i(A_j)$ and $g_i(A_k)$ of the alternatives A_j and A_k respectively, and introducing threshold levels for the difference $g_i(A_j)-g_i(A_k)$, the decision maker may declare that he/she is indifferent between the alternatives under consideration, that he/she has a weak or a strict preference for one of the two, or that he/she is unable to express any of these preference relations. Therefore, the set of binary relations of alternatives, the so-called outranking relations, may be complete or incomplete. Next, the decision maker is requested to assign weights or importance factors to the criteria in order to express their relative importance.

Through a series of consecutive assessments of the outranking relations of the alternatives, ELECTRE elicits the so-called concordance index, defined as the amount of evidence to support the conclusion that A_j outranks, or dominates, A_k, as well as the discordance, the counter-part of concordance index.

Finally, the ELECTRE method yields a whole system of binary outranking relations between the alternatives. Because the system is not necessarily complete, the ELECTRE method is sometimes unable to identify the preferred alternative. It only produces a core of leading alternatives. This method has a clearer view of alternatives by eliminating less favorable ones, especially convenient while encountering few criteria with large number of alternatives in a decision making problem [Lootsma, 1990]. The organization of the ELECTRE method is best illustrated in the following steps [Benayoun, *et al.*, 1966]:

<u>Step 1. Normalizing the Decision Matrix</u>

This procedure transforms various units in the decision matrix into dimensionless comparable units by using the following equation:

$$x_{ij} = \frac{a_{ij}}{\sqrt{\sum_{i=1}^{M} a_{ij}^2}}.$$

Therefore, the normalized matrix **X** is defined as follows:

$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1N} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2N} \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ x_{M1} & x_{M2} & x_{M3} & \cdots & x_{MN} \end{bmatrix},$$

where *M* is the number of alternatives and *N* is the number of criteria, and x_{ij} is the new and dimensionless preference measure of the *i*-th alternative in terms of the *j*-th criterion.

<u>Step 2. Weighting the Normalized Decision Matrix</u>

The column of the **X** matrix is then multiplied by its associated weights which were assigned to the criteria by the decision maker. Therefore, the weighted matrix, denoted as **Y**, is:

$$\mathbf{Y} = \mathbf{X}W,$$

where:

$$\mathbf{Y} = \begin{bmatrix} y_{11} & y_{12} & y_{13} & \cdots & y_{1N} \\ y_{21} & y_{22} & y_{23} & \cdots & y_{2N} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ y_{M1} & y_{M2} & y_{M3} & \cdots & y_{MN} \end{bmatrix} = \begin{bmatrix} w_1 x_{11} & w_2 x_{12} & w_3 x_{13} & \cdots & w_N x_{1N} \\ w_1 x_{21} & w_2 x_{22} & w_3 x_{23} & \cdots & w_N x_{2N} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ w_1 x_{M1} & w_2 x_{M2} & w_3 x_{M3} & \cdots & w_N x_{MN} \end{bmatrix},$$

and:

$$W = \begin{bmatrix} w_1 & 0 & 0 & \ldots & 0 \\ 0 & w_2 & 0 & \ldots & 0 \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ 0 & 0 & 0 & \ldots & w_M \end{bmatrix}, \quad and \quad also \quad \sum_{i=1}^{N} w_i = 1.$$

Step 3. Determine the Concordance and Discordance Sets

The concordance set C_{kl} of two alternatives A_k and A_l, where $M\$k$, $l \geq 1$, is defined as the set of all criteria for which A_k is preferred to A_l. That is, the following is true:

$C_{kl} = \{j, \text{ such that: } y_{kj} \geq y_{lj}\}$, for $j = 1, 2, 3, ..., N$.

The complementary subset is called the discordance set and it is described as follows:

$D_{kl} = \{j, \text{ such that: } y_{kj} < y_{lj}\}$, for $j = 1, 2, 3, ..., N$.

Step 4. Construct the Concordance and Discordance Matrices

The relative value of the elements in the concordance matrix **C** is calculated by means of the concordance index. The concordance index c_{kl} is the sum of the weights associated with the criteria contained in the concordance set. That is, the following is true:

$$C_{kl} = \sum_{j \in C_{kl}} w_j, \quad for \quad j = 1, 2, 3, ..., N.$$

The concordance index indicates the relative importance of alternative A_k with respect to alternative A_l. Apparently, $0 \leq c_{kl} \leq 1$. Therefore, the concordance matrix **C** is defined as follows:

$$C = \begin{bmatrix} - & c_{12} & c_{13} & \cdots & c_{1M} \\ c_{21} & - & c_{23} & \cdots & c_{2M} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ c_{M1} & c_{M2} & c_{M3} & \cdots & - \end{bmatrix}.$$

It should be noted here that the entries of matrix **C** are not defined when $k = l$.

The discordance matrix **D** expresses the degree that a certain alternative A_k is worse than a competing alternative A_l. The elements $d_{k\,l}$ of the discordance matrix are defined as follows:

$$d_{k\,l} \;=\; \frac{\max\limits_{j \in D_{k\,l}} |y_{kj} - y_{lj}|}{\max\limits_{j} |y_{kj} - y_{lj}|}. \tag{4-6}$$

The discordance matrix is defined as follows:

$$D \;=\; \begin{bmatrix} - & d_{12} & d_{13} & \dots & d_{1M} \\ d_{21} & - & d_{23} & \dots & d_{2M} \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ d_{M1} & d_{M2} & d_{M3} & \dots & - \end{bmatrix}.$$

As before, the entries of matrix **D** are not defined when $k = l$.

It should also be noted here that the previous two $M \times M$ matrices are **<u>not symmetric</u>**.

<u>Step 5. Determine the Concordance and Discordance Dominance Matrices</u>

The concordance dominance matrix is constructed by means of a threshold value for the concordance index. For example, A_k will only have a chance to dominate A_l if its corresponding concordance index c_{kl} exceeds at least a certain threshold value $\underline{c}$. That is, the following is true:

$$c_{kl} \;\geq\; \underline{c},$$

The threshold value $\underline{c}$ can be determined as the average concordance index. That is, the following relation is true:

$$\underline{c} \;=\; \frac{1}{M(M-1)} \times \sum_{\substack{k=1 \\ \text{and } k \neq l}}^{M} \; \sum_{\substack{l=1 \\ \text{and } l \neq k}}^{M} c_{k\,l}. \tag{4-7}$$

Based on the threshold value, the concordance dominance matrix **F** is determined as follows:

$f_{kl} = 1$, if $c_{kl} \geq \underline{c}$,
$f_{kl} = 0$, if $c_{kl} < \underline{c}$.

Similarly, the discordance dominance matrix **G** is defined by using a threshold value $\underline{d}$, where $\underline{d}$ is defined as follows:

$$\underline{d} \;=\; \frac{1}{M(M-1)} \sum_{\substack{k=1 \\ \text{and } k \neq l}}^{M} \; \sum_{\substack{l=1 \\ \text{and } l \neq k}}^{M} d_{kl}, \tag{4-8}$$

and: $g_{kl} = 1$, if $d_{kl} \geq \underline{d}$,
$g_{kl} = 0$, if $d_{kl} < \underline{d}$.

<u>Step 6. Determine the Aggregate Dominance Matrix</u>

The elements of the aggregate dominance matrix **E** are defined as follows:

$$e_{kl} = f_{kl} \times g_{kl}. \tag{4-9}$$

<u>Step 7. Eliminate the Less Favorable Alternatives</u>

From the aggregate dominance matrix, we could get a partial-preference ordering of the alternatives. If e_{kl}

= 1, then this means that A_k is preferred to A_l by using both concordance and discordance criteria.

If any column of the aggregate dominance matrix has at least one element equal to 1, this column is *"ELECTREally"* dominated by the corresponding row. Therefore, we simply eliminate any column(s) which have an element equal to 1. Then, the best alternative is the one which dominates all other alternatives in this manner.

4.2.6 The TOPSIS Method

TOPSIS *(the Technique for Order Preference by Similarity to Ideal Solution)* was developed by Hwang and Yoon [1981] as an alternative to the ELECTRE method. The basic concept of this method is that the selected alternative should have the shortest distance from the ideal solution and the farthest distance from the negative-ideal solution in a geometrical sense.

TOPSIS assumes that each attribute has a tendency of monotonically increasing or decreasing utility. Therefore, it is easy to locate the ideal and negative-ideal solutions. The Euclidean distance approach is used to evaluate the relative closeness of alternatives to the ideal solution. Thus, the preference order of alternatives is yielded through comparing these relative distances.

The TOPSIS method evaluates the following decision matrix which refers to M alternatives which are evaluated in terms of N criteria:

$$D = \begin{bmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1N} \\ x_{21} & x_{22} & x_{23} & \dots & x_{2N} \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ x_{M1} & x_{M2} & x_{M3} & \dots & x_{MN} \end{bmatrix}.$$

where x_{ij} denotes the performance measure of the i-th alternative in terms of the j-th criterion. For a clear view of this method, the TOPSIS method is presented next as a series of successive steps.

Step 1. Construct the Normalized Decision Matrix

This process tries to convert the various attribute dimensions into nondimensional attributes similarly as with the ELECTRE method. An element r_{ij} of the normalized decision matrix **R** can be calculated as follows:

$$r_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1}^{M} x_{ij}^2}}. \qquad (4\text{-}10)$$

Step 2. Construct the Weighted Normalized Decision Matrix

A set of weights $W = (w_1, w_2, w_3, \dots, w_N)$, (where: $\Sigma w_i = 1$) defined by the decision maker is accommodated to the decision matrix to generate the weighted normalized matrix **V** as follows:

$$V = \begin{bmatrix} w_1 r_{11} & w_2 r_{12} & w_3 r_{13} & \dots & w_N r_{1N} \\ w_1 r_{21} & w_2 r_{22} & w_3 r_{23} & \dots & w_N r_{2N} \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ w_1 r_{M1} & w_2 r_{M2} & w_3 r_{M3} & \dots & w_N r_{MN} \end{bmatrix}.$$

<u>Step 3. Determine the Ideal and the Negative-ideal Solutions</u>

The ideal A^* and the negative-ideal A^- solutions are defined as follows:

$$A^* = \{ (\max_i v_{ij} \mid j \in J), (\min_i v_{ij} \mid j \in J^/) \mid i = 1,2,3, ..., M\} = \{ v_{1*}, v_{2*}, ..., v_{N*} \}. \quad (4\text{-}11)$$

$$A^- = \{ (\min_i v_{ij} \mid j \in J), (\max_i v_{ij} \mid j \in J^/) \mid i = 1,2,3, ..., M\} = \{ v_{1}\text{-}, v_{2}\text{-}, ..., v_{N}\text{-} \}. \quad (4\text{-}12)$$

where: $J = \{j = 1,2,3, ..., N \mid j$ associated with benefit criteria$\}$,
$J^/= \{j = 1,2,3, ..., N \mid j$ associated with cost criteria$\}$.

For the benefit criteria, the decision maker wants to have a maximum value among the alternatives. For the cost criteria, the decision maker wants to have a minimum value among alternatives. Obviously, A^* indicates the most preferable alternative or ideal solution. Similarly, A^- indicates the least preferable alternative or negative-ideal solution.

<u>Step 4. Calculate the Separation Measure</u>

The N-dimensional Euclidean distance method is next applied to measure the separation distances of each alternative to the ideal solution and negative-ideal solution.

$$S_{i*} = (\Sigma(v_{ij} - v_{j*})^2)^{1/2}, \quad i = 1,2,3, ..., M, \quad (4\text{-}13)$$

where S_{i*} is the separation (in the Euclidean sense) of each alternative from the ideal solution.

$$S_{i-} = (\Sigma(v_{ij} - v_{j-})^2)^{1/2}, \quad i = 1,2,3, ..., M, \quad (4\text{-}14)$$

where S_{i-} is the separation (in the Euclidean sense) of each alternative from the negative-ideal solution.

<u>Step 5. Calculate the Relative Closeness to the Ideal Solution</u>

The relative closeness of an alternative A_i with respect to the ideal solution A^* is defined as follows:

$$C_{i*} = S_{i-} / (S_{i*} + S_{i-}), \; 0 \le C_{i*} \le 1, \; i = 1,2,3, ..., M. \quad (4\text{-}15)$$

Apparently, $C_{i*} = 1$, if $A_i = A^*$, and $C_{i-} = 0$, if $A_i = A^-$.

<u>Step 6. Rank the Preference Order</u>

The best satisfied alternative can now be decided according to preference rank order of C_{i*}. Therefore, the best alternative is the one that has the shortest distance to the ideal solution. The relationship of alternatives reveals that any alternative which has the shortest distance to the ideal solution is guaranteed to have the longest distance to the negative-ideal solution.

5 Sensitivity Analysis of MCDM Methods

As it was stated earlier, often data in MCDM problems are difficult to be quantified or are easily changeable. Thus, often the decision maker needs to first estimate the data with some accuracy, and later estimate more critical data with higher accuracy. In this way, the decision maker can rank the alternatives with high confidence and not overestimate non critical data. The above considerations lead to the need of performing a sensitivity analysis on a MCDM problem.

The objective of a typical sensitivity analysis of an MCDM problem is to find out when the input data (i.e., the a_{ij} and w_j values) are changed into new values, how the ranking of the alternatives will change. In the literature there has been some discussion on how to perform a sensitivity analysis in MCDM. Insua [1990] demonstrated that decision making problems may be remarkably sensitive to some reasonable variations in the parameters of the problems. His conclusion justified the necessity of sensitivity analysis in MCDM. Evans [1984], explored a linear programming -like sensitivity analysis in the decision making problems consisting of a single set of decision alternatives and states of nature. In his method, the optimal alternative is represented as a bounded convex polyhedron in the probability state space. Using the geometric characteristics of the optimal regions, he defined the confidence sphere of the optimal alternatives. The larger the confidence sphere, the less sensitive the optimal alternative will be to the state probabilities.

Masuda [1990] studied some sensitivity issues of the AHP method. In his paper, he focused on how changes on entire columns of the decision making matrix may affect the values of the composite priorities of the alternatives. In his method, he generated the sensitivity coefficient of the final priority vector of the alternatives to each of the column vectors in the decision matrix. A large coefficient means that the values of the final priorities of the alternatives will change more greatly if there is a slight change in the corresponding column vector of the decision matrix. However, that does not guarantee that a ranking reversal among the alternatives due to the change of the column vectors is sure to happen. Finally, Triantaphyllou and Sanchez [1997] proposed a unified approach for a sensitivity analysis for three major MCDM methods. These methods are: the WSM, the WPM and the AHP (original and revised). Their approach examines the effect of the changes of a single parameter (i.e., an a_{ij} or w_j value) on the final rankings of the alternatives. That approach can be seen as an extension of Masuda's method with its focus on the ranking reversal of the alternatives which is more useful in practical applications. Also in that paper, the authors have done some empirical studies to determine the most critical criterion (w_j) as well as the most critical performance value (a_{ij}) in a general MCDM problem.

Sensitivity analysis is a fundamental concept for the effective use and implementation of quantitative decision models [Dantzig, 1963]. It is just too important to be ignored in the application of an MCDM method to a real life problem.

6 Data Estimation for MCDM Problems

One of the most crucial steps in many decision making methods is the accurate estimation of the pertinent data. This problem is particularly crucial in methods which need to elicit qualitative information from the decision maker. Very often qualitative data cannot be known in terms of absolute values. For instance, *what is the worth of the i-th alternative in terms of a political impact criterion?* Although information about questions like the previous one may be vital in making the correct decision, it is very difficult, if not impossible, to quantify it correctly. Therefore, many decision making methods attempt to determine the **relative** importance, or weight, of the alternatives in terms of each criterion involved in a given decision making problem.

An approach based on pairwise comparisons which was proposed by Saaty (see, for instance, [Saaty, 1980 and 1983]) has long attracted the interest of many researchers. Pairwise comparisons are used to determine the relative importance of each alternative in terms of each criterion. In this approach a decision maker has to express his/her opinion about the value of one single pairwise comparison at a time. Usually, the decision maker has to choose his/her answer among 10-17 discrete choices. Each choice is a linguistic phrase. Some examples of such linguistic phrases are: *"A is more important than B"*, or *"A is of the same importance as B"*, or *"A is a little more important than B"*, and so on. The focus here is not on the wording of these linguistic statements, but, instead, on the numerical values which should be associated with such statements.

The main problem with the pairwise comparisons is **how to quantify** the linguistic choices selected by the decision maker during their evaluation. All the methods which use the pairwise comparisons approach eventually express the qualitative answers of a decision maker into some numbers which, most of the time, are ratios of integers. A case in which pairwise comparisons are expressed as differences (instead of ratios) was used to define similarity relations and is described by Triantaphyllou in [1993]. The next section examines the issue of quantifying pairwise comparisons. Since pairwise comparisons are the keystone of these decision making processes, correctly quantifying them is the most crucial step in multi-criteria decision making methods which use qualitative data.

Many of the previous problems are not bound only to the AHP. They are present with any method which has to elicit information from pairwise comparisons. These problems can be divided into the following three categories:

(i) How to quantify the pairwise comparisons.
(ii) How to process the resulted reciprocal matrices.
and (iii) How to process the decision matrices.

Next we consider some of the main ideas related with pairwise comparisons. In the sub-sections that follow, we consider each one of the previous challenges, and discuss some remedies which have been proposed.

6.1 *Problem #1:* On the Quantification of Pairwise Comparisons

Pairwise comparisons are quantified by using a **scale**. Such a scale is an one-to-one mapping between the set of discrete linguistic choices available to the decision maker and a discrete set of numbers which represent the importance, or weight, of the previous linguistic choices. There are two major approaches in developing such scales. The first approach is based on the linear scale proposed by Saaty [1980] as part of the AHP. The second approach was proposed by Lootsma in [1988 and 1990] and in [Lootsma, *et al.*, 1990] and determines **exponential scales**. Both approaches depart from some psychological theories and develop the numbers to be used based on these psychological theories.

6.1.1 Scales Defined on the Interval [9, 1/9]

In 1846 Weber stated his law regarding a stimulus of measurable magnitude. According to his law a change in sensation is noticed if the stimulus is increased by a constant percentage of the stimulus itself [Saaty, 1980]. That is, people are unable to make choices from an infinite set. For example, people cannot distinguish between two very close values of importance, say 3.00 and 3.02. Psychological experiments have also shown that individuals cannot simultaneously compare more than seven objects (plus or minus two) [Miller, 1956]. This is the main reasoning used by Saaty to establish 9 as the upper limit of his scale, 1 as the lower limit and a unit difference between successive scale values.

The values of the pairwise comparisons are determined according to the scale introduced by Saaty [1980]. According to this scale (which we call Scale1), the available values for the pairwise comparisons are members of the set: {9, 8, 7, 6, 5, 4, 3, 2, 1, 1/2, 1/3, 1/4, 1/5, 1/6, 1/7, 1/8, 1/9}. The above numbers illustrate that the values for the pairwise comparisons can be grouped into the two intervals [9, 1] and [1, 1/9]. As it was stated above, the values in the interval [9, 1] are **evenly distributed**, while the values in the interval [1, 1/9] are **skewed** to the right end of this interval.

There is no good reason why for a scale defined on the interval [9, 1/9] the values on the sub-interval [9, 1] should be evenly distributed. An alternative scale could have the values evenly distributed in the interval [1, 1/9], while the values in the interval [9, 1] could be simply the reciprocals of the values in the interval [1, 1/9]. This consideration leads to the scale (which we call Scale2) with the following values: {9, 9/2, 9/3, 9/4, 9/5, 9/6, 9/7, 9/8, 1, 8/9, 7/9, 6/9, 5/9, 4/9, 3/9, 2/9, 1/9}. This scale was originally presented by Ma and Zheng in [1991]. In the second scale each successive value on the interval [1, 1/9] is (1 - 1/9) / 8 = 1/9 units apart. In this way, the values in the interval [1, 1/9] are **evenly distributed**, while the values in [9, 1] are simply the reciprocals of the values in [1, 1/9]. It should be stated here that the notion of having a scale with a group of values evenly distributed, is followed in order to be in agreement with the same characteristic of the original Saaty scale. As it will be seen in the next section, other scales can be defined without having evenly distributed values.

Besides the second scale, many other scales can be generated. One way to generate new scales is to consider **weighted** versions between the previous two scales. That is, for the interval [1, 1/9] the values can be calculated using the formula:

NewValue = Value(Scale1) + (Value(Scale2) - Value(Scale1))*(α/100).

In the previous formula the values of α can range from 0 to 100. Then, the values in the interval [9, 1] are the **reciprocals** of the above values. For α = 0 Scale1 is derived, while for α = 100 Scale2 is derived.

6.1.2 Exponential Scales

A class of exponential scales has been introduced by Lootsma [1988 and 1990] and [Lootsma, *et al.*, 1990]. The development of these scales is based on an observation in psychology about stimulus perception (denoted as e_i). According to that observation, due to Roberts [1979], the difference $e_{n+1} - e_n$ must be greater than or equal to the smallest perceptible difference, which is proportional to e_n. As a result of Robert's observation the numerical equivalents of these linguistics choices need to satisfy the following relations:

$$e_{n+1} - e_n = \epsilon\, e_n, \quad (\text{where } \epsilon > 0) \quad \text{or:}$$
$$e_{n+1} = (1+\epsilon)\, e_n = (1+\epsilon)^2\, e_{n-1} = \ldots$$
$$\ldots = (1+\epsilon)^{n+1}\, e_o, \quad (\text{where: } e_o = 1) \quad \text{or:} \quad e_n = e^{\gamma \times n}.$$

In the previous expressions the parameter γ is **unknown** (or, equivalently, ϵ is unknown), since $\gamma = ln(1+\epsilon)$, and e is the basis of the natural logarithms (please note that e_i is just the notation of a variable).

Another difference between exponential scales and the Saaty scale is the number of categories allowed by the exponential scales. There are only four major linguistically distinct categories, plus three so-called threshold categories between them. The threshold categories can be used if the decision maker hesitates between the main categories. For a more detailed documentation on psychophysics we refer the reader to Marks [1974], Michon, *et al.*, [1769], Roberts [1979], Zwicker [1982], and Stevens and Hallowell Davis [1983]. The reader will find that the sensory systems for the perception of tastes, smells, and touches follow the power law with exponents near 1.

6.1.3 Evaluating Different Scales

In order for different scales to be evaluated, two evaluative criteria were developed by Triantaphyllou, *et al.*, in [1994]. Furthermore, a special class of pairwise matrices was also developed. These special matrices were then used in conjunction with the two evaluative criteria in order to investigate some stability properties of different scales.

The most important observation of that study is that the results illustrate very clearly that **there is no single scale which is the best scale for all cases**. Similarly, the results illustrate that there is no single scale which is the worst scale for all cases. However, according to these computational results, the best (or worst) scale can be determined only if the number of the alternatives is known and the relative importance of the weights of the two evaluative criteria has been assessed.

6.2 *Problem #2:* Processing Reciprocal Matrices with Pairwise Comparisons

At this point it is assumed that the decision maker has determined the values of all the pairwise comparisons. That is, available are the values a_{ij} (for i,j = 1,2,3,...,N), where a_{ij} represents the relative performance of alternative A_i when it is compared with alternative A_j in terms of a single criterion. Given these values, the decision maker needs to determine the relative weights, say W_i (i = 1,2,3,...,N), of the alternatives in terms of the single criterion. Saaty [1980] has proposed a method which asserts that the desired weights are the elements of the right principal eigenvector of the matrix with the pairwise comparisons. This method has been evaluated under a continuity assumption by Triantaphyllou and Mann in [1990]. Moreover, other authors have proposed alternative approaches.

For instance, Chu, *et al.*, in [1979] observed that, given the data a_{ij}, the values W_i to be estimated are desired to have the following property:

$$a_{ij} \approx W_i/W_j.$$

This is reasonable, since a_{ij} is meant to be the estimate of the ratio W_i/W_j. Then, in order to get the estimates for the W_i given the data a_{ij}, they proposed the following constrained optimization problem:

$$\text{minimize} \quad S = \sum_{i=1}^{N} \sum_{j=1}^{N} (a_{ij} W_j - W_i)^2$$

$$\text{subject to:} \quad \sum_{i=1}^{N} W_i = 1,$$

$$\text{and } W_i > 0, \quad \text{for any } i = 1,2,3,...,N.$$

They also gave an alternative expression S_1 that is more difficult to solve numerically. Specifically, they proposed:

$$\text{minimize} \quad S_1 = \sum_{i=1}^{N} \sum_{j=1}^{N} (a_{ij} - W_j / W_i)^2.$$

In Federov, *et al.*, [1982], a variation of the previous least-squares formulation was proposed. For the case of only one decision maker the authors recommended to use the following models:

$$\log a_{ij} = \log W_i - \log W_j + \Psi_1(W_i, W_j)\varepsilon_{ij},$$
$$\text{and} \quad a_{ij} = W_i / W_j + \Psi_2(W_i, W_j)\varepsilon_{ij},$$

where W_i and W_j are the true (and thus unknown) weights; $\Psi_1(X,Z)$ and $\Psi_2(X,Z)$ are given positive functions (where $X,Z > 0$). The random errors ε_{ij} are assumed to be independent with zero mean and unit variance. However, they fail to give a way of selecting the appropriate two previous positive functions.

In the following paragraphs we present the main idea which was originally described in Triantaphyllou, *et al.*, [1990]. In that treatment the assumption of the **human rationality** is made. According to that assumption the decision maker is a rational person. Rational persons are defined here as individuals who try to minimize their regret [Simon, 1961], to minimize losses, or to maximize profit [Write and Tate, 1973]. In the present context, minimization of regret of losses, or maximization of profit could be interpreted as the effort of the decision maker to minimize the errors involved in the pairwise comparisons.

As it was stated in the previous paragraphs, in the inconsistent case, the entry a_{ij} of matrix A is an estimate of the real ratio W_i/W_j. Since it is an estimate, the following is true:

$a_{ij} = (W_i/W_j)d_{ij}$, for $i, j = 1, 2, 3, ..., N$. (5-1)

In the previous relation, d_{ij} denotes the deviation of a_{ij} from being a perfectly accurate judgment. Obviously, if $d_{ij} = 1$, the a_{ij} value was perfectly estimated. From the previous formulation, we conclude that the errors involved in these pairwise comparisons are given by:

$\varepsilon_{ij} = d_{ij} - 1,$

or by using (5-1) above,

$\varepsilon_{ij} = a_{ij} (W_j/W_i) - 1.$ (5-2)

When the set of alternatives (or criteria) contains N elements, then $N(N\text{-}1)/2$ total pairwise comparisons need to be estimated. The corresponding $N(N\text{-}1)/2$ errors are (after using relations (5-1) and (5-2)):

$\varepsilon_{ij} = a_{ij} (W_j/W_i) - 1$, for $i, j = 1, 2, 3, ..., N$, and $j > 1$. (5-3)

Since the W_i's are relative weights which (in most cases) have to add up to 1, the following relation should also be satisfied:

$$\sum_{i=1}^{N} W_i = 1.00, \quad \text{and} \quad W_i > 0, \quad \text{for} \quad i = 1, 2, 3, ..., N. \qquad (5\text{-}4)$$

When the data (e.g., the pairwise comparisons) are perfectly consistent, then relations (5-3) and (5-4) can be written as follows:

$\mathbf{B} \times W = b.$ (5-5)

The vector b has zero entries everywhere, except that the last entry is equal to 1; the matrix **B** has the following structure (blank entries represent zeros):

$$
B = \left[\begin{array}{ccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots & N-1 & N & \\
-1 & a_{1,2} & & & & & & & & & 1 \\
-1 & & a_{1,3} & & & & & & & & 2 \\
-1 & & & a_{1,4} & & & & & & & 3 \\
\cdot & & & & \cdot & & & & & & \cdot \\
\cdot & & & & & \cdot & & & & & \cdot \\
\cdot & & & & & & \cdot & & & & \cdot \\
\cdot & & & & & & & \cdot & & & \cdot \\
\cdot & & & & & & & & a_{1,N-1} & & \cdot \\
-1 & & & & & & & & & a_{1,N} & N-1 \\
 & -1 & a_{2,3} & & & & & & & & 1 \\
 & -1 & & a_{2,4} & & & & & & & 2 \\
 & -1 & & & a_{2,5} & & & & & & 3 \\
 & \cdot & & & & \cdot & & & & & \cdot \\
 & \cdot & & & & & \cdot & & & & \cdot \\
 & \cdot & & & & & & \cdot & & & \cdot \\
 & \cdot & & & & & & & a_{2,N-1} & & \cdot \\
 & -1 & & & & & & & & a_{2,N} & N-2 \\
 & & & & & & & \cdot & & & \cdot \\
 & & & & & & & \cdot & & & \cdot \\
 & & & & & & & \cdot & & & \cdot \\
 & & & & & & & & -1 & a_{N-1,N} & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 &
\end{array}\right]
$$

The error minimization issue is interpreted in many cases (for instance, in regression analysis and in the linear least-squares problem) as the minimization of the sum of squares of the residual vector $r = b$ - $\mathbf{B} \times W$ [Stewart, 1973]. In terms of the previous formulation (5-5), this means that, in a real-life situation (i.e., when errors are not zero any more), the real intention of the decision maker is to minimize the following expression

$$f^2(x) = \| b - BW \|_2^2, \qquad (5\text{-}6)$$

which, apparently, expresses a typical linear least-squares problem.

In [Triantaphyllou, *et al.*, 1990] all the previous methods were tested in terms of an example originally presented by Saaty in [1977] and also later used by other authors (e.g., Chu, *et al.*, [1979] and [Federov, *et al.*, 1982]). In that test it was found that the proposed human rationality approach results in much smaller residuals. Moreover, in the same study it was found, on thousands of randomly generated test problems, that the eigenvalue approach may result in considerably higher residual values than the proposed least-squares approach which uses the previous human rationality assumption.

6.3 ***Problem #3:*** **Processing the Decision Matrices**

In Triantaphyllou and Mann [1989] the AHP, revised AHP, weighted sum model (WSM) [Fishburn, 1967], and the weighted product model (WPM) [Miller and Starr, 1969] were examined in terms of two evaluative criteria. That study focused on the last step of any MCDM method which involves the processing of the final decision matrix. That is, given the weights of relative performance of the decision criteria, and the performance of the alternatives in terms of each one of the decision criteria, then determine what is the ranking (or relative priorities) of the alternatives.

As it was shown in Triantaphyllou and Mann [1989], however, these methods can give different answers to the same problem. Since the truly best alternative is the same **regardless** of the method chosen, an estimation of the accuracy of each method is highly desirable. The most difficult problem that arises here is how one can evaluate a multi-dimensional decision making method when the true best alternative is not known. Two evaluative criteria were introduced in [Triantaphyllou and Mann, 1989] for the above purpose.

The **first evaluative criterion** has to do with the premise that ***a method which is accurate in multi-dimensional problems should also be accurate in single-dimensional problems. There is no reason for an accurate multi-dimensional method to fail in giving accurate results in single-dimensional problems, since single-dimensional problems are special cases of multi-dimensional ones.*** Because the first method, the WSM, gives the most acceptable results for the majority of single-dimensional problems, the result of the WSM was used as the standard for evaluating the other three methods in this context.

The **second evaluative criterion** considers the premise that ***a desirable method should not change the indication of the best alternative when an alternative (not the best) is replaced by another worse alternative (given that the importance of each criterion remains unchanged)***.

In Triantaphyllou and Mann [1989] the previous two evaluative criteria were applied on random test problems with the numbers of decision criteria and alternatives taking the values 3, 5, 7, ..., 21. In those experiments it was found that all the previous four MCDM methods were inaccurate. Furthermore, these results were used to form a decision problem in which the four methods themselves were the alternatives. The decision criteria were derived by considering the two evaluative criteria. To one's greatest surprise, one method would recommend another, rival method, as being the best method! However, the final results seemed to suggest that the **revised AHP** was the most efficient MCDM method of the ones examined. This was reported in Triantaphyllou and Mann [1989] as a **decision making paradox**. Finally, a different approach of evaluating the performance of the AHP and the revised AHP is described by Triantaphyllou and Mann in [1995]. In that treatment it was found that these two methods may yield dramatically inaccurate results (more than 80% of the time on all the problems).

7 Concluding Remarks

There is no doubt that many real life problems can be dealt with as MCDM problems. Although the mathematical procedures for processing the pertinent data are rather simple, the real challenge is in quantifying these data. This is a non trivial problem. In matter of fact, it is not even a well defined problem. For these reasons, the literature has an abundance of competing methods. The main problem is that often nobody can know what is the optimal alternative. Operations research provides a systematic framework for dealing with such problems.

This paper discussed some of the challenges facing practitioners and theoreticians in some of the methodological problems in MCDM theory. Although it is doubtful that the *"perfect"* MCDM approach will ever be found, it is always a prudent idea for the user to be aware of the main controversies in the field. Although the search for finding the best MCDM method may never end, research in this area of decision making is still critical and valuable.

REFERENCES

1. Belton, V., and T. Gear, "On a short-coming of Saaty's method of analytic hierarchies," *Omega*, 228-230, 1983.

2. Benayoun, R., B. Roy, and N. Sussman, "Manual de reference du programme electre, Note de Synthese et Formation," No. 25, *Direction Scientifique SEMA*, Paris, Franch, 1966.

3. Boucher, T.O. and E.L. McStravic, "Multi-attribute Evaluation Within a Present Value Framework and its Relation to the Analytic Hierarchy Process," *The Engineering Economist*, 37, 55-71, 1991.

4. Bridgman, P.W., *Dimensional Analysis,* Yale University Press, New Haven, CN, 1922.

5. Cambron, K.E. and G.W. Evans, "Layout Design Using the Analytic Hierarchy Process," *Computers & Industrial Engineering,* 20, 221-229, 1991

6. Chen, S.J. and C.L. Hwang, *Fuzzy Multiple Attribute Decision Making: Methods and Applications,* Lecture Notes in Economics and Mathematical Systems, No. 375, Sringer-Verlag, Berlin, Germany, 1992.

7. Chu, A.T.W., R.E. Kalaba, and K. Spingarn, "A comparison of two methods for determining the weights of belonging to fuzzy sets," *Journal of Optimization Theory and Applications*, 27, 531-538, 1979.

9. Dantzig, G.B., *Linear programming and extensions*. Princeton University Press, Princeton, NJ, 1963.

10. Evans, J.R. "Sensitivity analysis in decision theory," *Decision Sciences*, 1(15), 239-247, 1984.

11. Federov, V.V., V.B. Kuz'min, and A.I. Vereskov, "Membership degrees determination from Saaty matrix totalities," Institute for System Studies, Moscow, USSR. Paper appeared in: *Approximate Reasoning in Decision Analysis*, M.M. Gupta, and E. Sanchez (editors), North-Holland Publishing Company, 23-30, 1982.

12. Fishburn, P.C., *Additive Utilities with Incomplete Product Set: Applications to Priorities and Assignments*, Operations Research Society of America (ORSA) Publication, Baltimore, MD, 1967.

13. Hwang C.L. and K. Yoon, *Multiple Attribute Decision Making: Methods and Applications,* Springer-Verlag, New York, NY, 1981.

14. Kuhn, H.W. and A.W. Tucker, "Nonlinear Programming," *Proc. 2nd Berkeley Symp. Math. Stat. Prob.*, 481-492, 1951.

15. Lootsma, F.A., "Numerical scaling of human judgment in pairwise-comparison methods for fuzzy multi-criteria decision analysis," *Mathematical Models for Decision Support*. NATO ASI Series F, Computer and System Sciences, Springer-Verlag, Berlin, Germany, 48, 57-88, 1988.

16. Lootsma, F.A., T.C.A. Mensch, and F.A. Vos, "Multi-criteria analysis and budget reallocation in long-term research planning," *European Journal of Operational Research*, 47, 293-305, 1990.

17. Lootsma, F.A., "The French and the American school in multi- criteria decision analysis," *Recherche Operationnele / Operations Research*, 24(3), 263-285, 1990.

18. Ma, D. and X. Zheng, "9/9-9/1 scale method of the AHP," *Proceedings of the 2nd International Symposium on the AHP*, Vol. 1, Pittsburgh, PA, 197-202, 1991.

19. Marks, L.E., *Sensory Processes, The New Psychophysics*, Academic Press, New York, NY, 1974.

20. Masuda, T. "Hierarchical sensitivity analysis of the priorities used in the Analytic Hierarchy Process," *Systems Science*, 21(2), 415-427, 1990.

21. Michon, J.A., E.G.J. Eijkman, and L.F.W. de Klerk, *Handboek der Psychonomie*, (in Dutch), Van Loghum Slaterus, Deventer, The Netherlands, 1976.

22. Miller, D.W., and M.K. Starr, *Executive Decisions and Operations Research*, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1969.

23. Miller, G.A., "The magical number seven plus or minus two: some limits on our capacity for processing information," *Psychological Review*, 63, 81-97, 1956.

24. Putrus, P. "Accounting for Intangibles in Integrated Manufacturing (nonfinancial justification based on the Analytical Hierarchy Process)," *Information Strategy*, 6, 25-30, 1990.

25. Ríos Insua, D. *Sensitivity analysis in multi-objective decision making*. Lecture Notes in Economics and Mathematical Systems, Springer-Verlag, Berlin, Germany, 1990.

26. Roberts, F.S., *Measurement Theory*, Addison-Wesley, Reading, Mass, 1979.

27. Roy, B., "How the outranking relation helps multiple criteria decision making." In: *Multiple Criteria Decision Making*, Cochrane and Zeleny (Eds.), University of South Carolina Press, SC, 179-201, 1973.

28. Saaty, T.L., "A scaling method for priorities in hierarchical structures," *Journal of Mathematical Psychology*, 15, 57-68, 1977.

29. Saaty, T.L., *The Analytic Hierarchy Process*, McGraw-Hill International, New York, NY, 1980.

30. Saaty, T.L., "Axiomatic foundations of the Analytic Hierarchy process," *Management Science*, 32(2), 841-855, 1983.

31. Saaty, T.L., "An exposition of the AHP in reply to the paper 'Remarks on the Analytic Hierarchy Process'," *Management Science*, 36(3), 259-268, 1990.

32. Saaty, T.L., *Fundamentals of Decision Making and Priority Theory with the AHP*, RWS Publications, Pittsburgh, PA, U.S.A., 1994.

33. Simon, H.A., *Models of Man*, 2nd Edition, John Wiley and Sons, New York, NY, 1961.

34. Stevens, S.S. and M.D. Hallowell Davis, *Hearing, its Psychology and Physiology*, American Institute of Physics, New York, NY, 1983.

35. Stewart, S.M., *Introduction to Matrix Computations*, Academic Press, New York, NY, 1973.

36. Triantaphyllou, E., "A quadratic programming approach in estimating similarity relations," *IEEE Transactions on Fuzzy Systems*, 1(2), 138-145, 1993.

37. Triantaphyllou, E., F.A. Lootsma, P.M. Pardalos, and S.H. Mann, "On the Evaluation and Application of Different Scales For Quantifying Pairwise Comparisons in Fuzzy Sets," *Multi-Criteria Decision Analysis*, 3, 1-23, 1994.

38. Triantaphyllou, E., and S.H. Mann, "An evaluation of the AHP and the revised AHP when the eigenvalue method is used under a continuity assumption," *Computers and Industrial Engineering*, 2(1), 35-44, 1995.

39. Triantaphyllou, E., and S.H. Mann, "An evaluation of the eigenvalue approach for determining the membership values in fuzzy sets," *Fuzzy Sets and Systems*. 35(3), 295-301, 1990.

40. Triantaphyllou, E., and S.H. Mann, "An examination of the effectiveness of multi-dimensional decision-making methods: A decision-making paradox," *International Journal of Decision Support Systems*, 5, 303-312, 1989.

41. Triantaphyllou, E., P.M. Pardalos, and S.H. Mann, "A minimization approach to membership evaluation in fuzzy sets and error analysis," *Journal of Optimization Theory and Applications*, 66(2), 275-287, 1990.

42. Triantaphyllou, E., P.M. Pardalos, and S.H. Mann, "The Problem of Evaluating Membership Values in Real World Situations," *Operations Research and Artificial Intelligence: The Integration of Problem Solving Strategies*, (D.E. Brown and C.C. White III, editors), Kluwer Academic Publishers, 197-214, 1990.

43. Triantaphyllou, E. and A. Sanchez, "A Sensitivity Analysis Approach for Some Deterministic Multi-Criteria Decision-Making Methods," *Decision Sciences*, 28(1), 151-194, 1997.

44. Wabalickis, R.N., "Justification of FMS With the Analytic Hierarchy Process," *Journal of Manufacturing Systems*, 17, 175-182, 1988.

45. Wang, L., and T. Raz, "Analytic Hierarchy Process Based on Data Flow Problem," *Computers & Industrial Engineering*, 20, 355-365, 1991.

46. Write, C., and M.D. Tate, *Economics and Systems Analysis: Introduction for Public Managers*, Addison-Wesley, Reading, MA, 1973.

47. Zimmermann, H.-J., *Fuzzy Set Theory and Its Applications*, Kluwer Academic Publishers, Second Edition, Boston, MA, 1991.

48. Zwicker, E., *Psychoakustic*, Springer-Verlag, Berlin, Germany, 1982.