هو الحق

Fuzzy arithmetic

The Extension Principle

One of the most basic concepts of fuzzy set theory is the *extension principle.* Let $X_1 \times X_2 \times \ldots \times X_n$ be a universal product set and F a functional mapping of the form

$$F : X_1 \times X_2 \times \ldots \times X_n \mapsto Z \ , \tag{1.149}$$

$\widetilde{A}_2 \subseteq X_2$, ..., $\widetilde{A}_n \subseteq X_n$ be n fuzzy sets, defined by the membership functions $\mu_{\widetilde{A}_1}(x_1)$, $\mu_{\widetilde{A}_2}(x_2)$, ..., $\mu_{\widetilde{A}_n}(x_n)$, $x_i \in X_i$, $i = 1, 2, \ldots, n$. Then the membership function $\mu_{\widetilde{B}}(z)$, $z \in Z$, of the fuzzy set $\widetilde{B} \subseteq Z$ with

$$\widetilde{B} = F\big(\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_n\big) \tag{1.150}$$

is defined by

$$\mu_{\widetilde{B}}(z) = \begin{cases} \sup\limits_{z=F(x_1,x_2,\ldots,x_n)} \min\left\{\mu_{\widetilde{A}_1}(x_1), \mu_{\widetilde{A}_2}(x_2), \ldots, \mu_{\widetilde{A}_n}(x_n)\right\} \\ \qquad\qquad\qquad \text{if} \quad \exists\, z = F(x_1, x_2, \ldots, x_n) \\ 0 \qquad\qquad\qquad \text{otherwise} \ . \end{cases} \tag{1.151}$$

$$\widetilde{B} = F(\widetilde{A}) \tag{1.152}$$

is defined by

$$\mu_{\widetilde{B}}(z) = \begin{cases} \sup\limits_{z=F(x)} \mu_{\widetilde{A}}(x) & \text{if} \quad \exists\, z = F(x) \\ 0 & \text{otherwise} . \end{cases} \tag{1.153}$$

Example 1.10. As an example for the special case $n = 1$, let us consider the discrete fuzzy set $\widetilde{A}$, defined on the universal set $X = \mathbb{Z}$ of integer numbers by

$$\widetilde{A} = \{(-1, 0.1), (0, 0.4), (1, 1.0), (2, 0.4), (3, 0.1)\} \ . \tag{1.154}$$

The evaluation of the functional mapping F, defined for the crisp argument x by

$$z = F(x) = x^2 + 1 \ , \tag{1.155}$$

then results in the discrete fuzzy set $\widetilde{B}$, given on the universal set $Z = \mathbb{N}$ by

$$\widetilde{B} = F(\widetilde{A}) = \widetilde{A}^2 + 1 = \{(1, 0.4), (2, 1.0), (5, 0.4), (10, 0.1)\} \ . \tag{1.156}$$

$$\begin{aligned} \mu_{\widetilde{B}}(z = 2) &= \sup_{F(x)=2} \mu_{\widetilde{A}}(x) \\ &= \max\left[\mu_{\widetilde{A}}(x = -1), \mu_{\widetilde{A}}(x = 1)\right] \\ &= \max\left[0.1, 1.0\right] \\ &= 1.0 \ . \end{aligned} \tag{1.157}$$

Example 1.11. As an example for the case $n > 1$, let us consider the $n = 2$ discrete fuzzy sets $\widetilde{A}_1$ and $\widetilde{A}_2$, defined on the universal sets $X_1 = X_2 = \mathbb{Z}$ of integer numbers by

$$\widetilde{A}_1 = \{(-1, 0.1), (0, 0.4), (1, 1.0), (2, 0.5), (3, 0.1)\} \ , \tag{1.158}$$

$$\widetilde{A}_2 = \{(0, 0.2), (1, 0.4), (2, 1.0), (5, 0.4), (10, 0.1)\} \ . \tag{1.159}$$

The evaluation of the functional mapping F, defined for the crisp arguments x_1 and x_2 by

$$z = F(x_1, x_2) = x_1 + \frac{1}{2}\, x_2 \ , \tag{1.160}$$

$x_1^{<\mu_{\tilde{A}_1}(x_1)>}$ \ $x_2^{<\mu_{\tilde{A}_2}(x_2)>}$	$0^{<0.2>}$	$1^{<0.4>}$	$2^{<1.0>}$	$5^{<0.4>}$	$10^{<0.1>}$
$-1^{<0.1>}$	$-1^{<0.1>}$	$-0.5^{<0.1>}$	$0^{<0.1>}$	$1.5^{<0.1>}$	$4^{<0.1>}$
$0^{<0.4>}$	$0^{<0.2>}$	$0.5^{<0.4>}$	$1^{<0.4>}$	$2.5^{<0.4>}$	$5^{<0.1>}$
$1^{<1.0>}$	$1^{<0.2>}$	$1.5^{<0.4>}$	$2^{<1.0>}$	$3.5^{<0.4>}$	$6^{<0.1>}$
$2^{<0.5>}$	$2^{<0.2>}$	$2.5^{<0.4>}$	$3^{<0.5>}$	$4.5^{<0.4>}$	$7^{<0.1>}$
$3^{<0.1>}$	$3^{<0.1>}$	$3.5^{<0.1>}$	$4^{<0.1>}$	$5.5^{<0.1>}$	$8^{<0.1>}$
	$z^{<\min[\mu_{\tilde{A}_1}(x_1),\mu_{\tilde{A}_2}(x_2)]>},\ z = x_1 + \frac{1}{2}x_2$				

Result for the example :

$z = x_1 + \frac{1}{2}x_2$	$\min\left[\mu_{\tilde{A}_1}(x_1), \mu_{\tilde{A}_2}(x_2)\right]$		max
−1	0.1		0.1
−0.5	0.1		0.1
0	0.1	0.2	0.2
0.5	0.4		0.4
1	0.4	0.2	0.4
1.5	0.1	0.4	0.4
2	1.0	0.2	1.0
2.5	0.4	0.4	0.4
3	0.5	0.1	0.5
3.5	0.4	0.1	0.4
4	0.1	0.1	0.1
4.5	0.4		0.4
5	0.1		0.1
5.5	0.1		0.1
6	0.1		0.1
7	0.1		0.1
8	0.1		0.1
z			$\mu_{\tilde{B}}(z)$

Fuzzy numbers

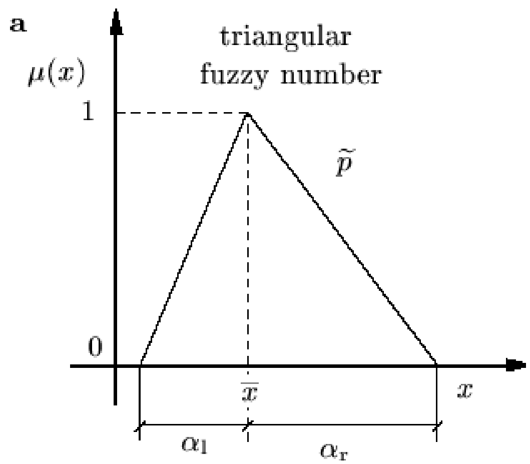
Triangular Fuzzy Number (Linear Fuzzy Number)

Due to its rather simple membership function of the linear type, the *triangular fuzzy number* or *linear fuzzy number* is one of the most frequently used fuzzy numbers. As an abbreviated form, we can introduce the notation

$$\widetilde{p} = \mathrm{tfn}(\overline{x}, \alpha_\mathrm{l}, \alpha_\mathrm{r}) \tag{2.5}$$

to define a triangular fuzzy number $\widetilde{p} \in \widetilde{\mathcal{P}}'(\mathbb{R})$ with the membership function

$$\mu_{\widetilde{p}}(x) = \begin{cases} 0 & \text{for} \quad x \leq \overline{x} - \alpha_\mathrm{l} \\ 1 + (x - \overline{x})/\alpha_\mathrm{l} & \text{for} \quad \overline{x} - \alpha_\mathrm{l} < x < \overline{x} \\ 1 - (x - \overline{x})/\alpha_\mathrm{r} & \text{for} \quad \overline{x} \leq x < \overline{x} + \alpha_\mathrm{r} \\ 0 & \text{for} \quad x \geq \overline{x} + \alpha_\mathrm{r} \end{cases} \qquad \text{or} \tag{2.6}$$



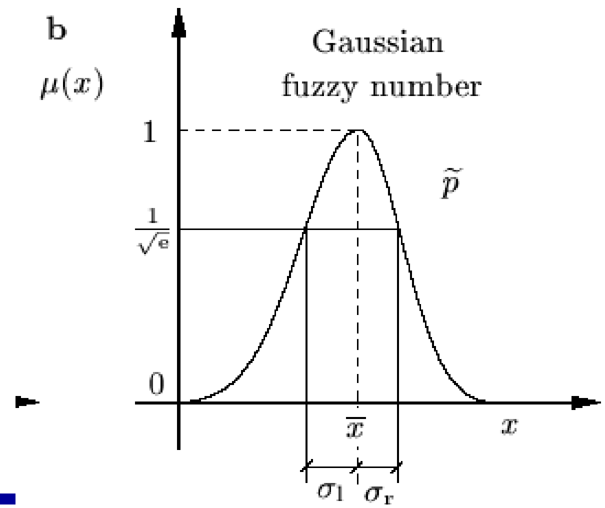
Gaussian Fuzzy Number

Another important type of fuzzy number is the *Gaussian fuzzy number*, where the membership function is characterized by a normalized and, in general, asymmetrically parameterized Gaussian function. We can introduce an abbreviated notation of the form

$$\widetilde{p} = \mathrm{gfn}(\overline{x}, \sigma_\mathrm{l}, \sigma_\mathrm{r}) \tag{2.9}$$

to define a Gaussian fuzzy number $\widetilde{p} \in \widetilde{\mathcal{P}}'(\mathbb{R})$ with the membership function

$$\mu_{\widetilde{p}}(x) = \begin{cases} \exp\left[-(x-\overline{x})^2/(2\,\sigma_\mathrm{l}^2)\right] & \text{for} \quad x < \overline{x} \\ \exp\left[-(x-\overline{x})^2/(2\,\sigma_\mathrm{r}^2)\right] & \text{for} \quad x \geq \overline{x} \end{cases} \qquad \forall\, x \in \mathbb{R}\,.$$



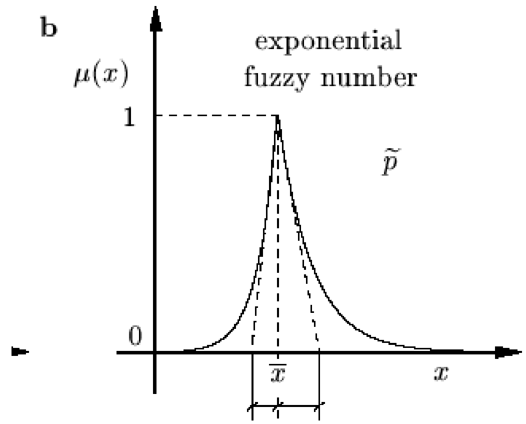
Exponential Fuzzy Number

The membership function of the *exponential fuzzy number* is of an exponential type, and we can introduce the abbreviated notation

$$\widetilde{p} = \mathrm{efn}(\overline{x}, \tau_{\mathrm{l}}, \tau_{\mathrm{r}}) \qquad (2.17)$$

to define the fuzzy number $\widetilde{p} \in \widetilde{\mathcal{P}}'(\mathbb{R})$ with the membership function

$$\mu_{\widetilde{p}}(x) = \begin{cases} \exp\left[-(x-\overline{x})/\tau_{\mathrm{l}} \right] & \text{for} \quad x < \overline{x} \\ \exp\left[-(x-\overline{x})/\tau_{\mathrm{r}} \right] & \text{for} \quad x \geq \overline{x} \end{cases} \quad \forall\, x \in \mathbb{R}\,. \qquad (2.18)$$



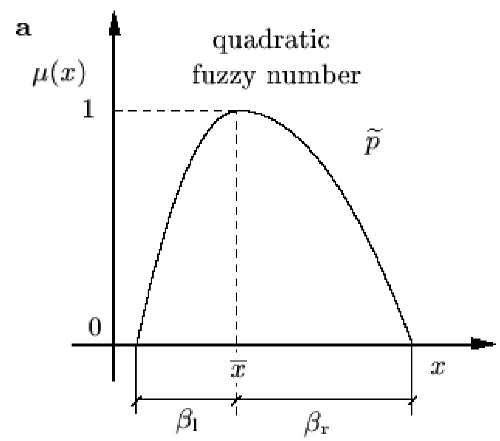
Quadratic Fuzzy Number

For the definition of the *quadratic fuzzy number*, we can introduce the abbreviated notation

$$\widetilde{p} = \mathrm{qfn}(\overline{x}, \beta_{\mathrm{l}}, \beta_{\mathrm{r}}) \;, \tag{2.14}$$

which leads to a fuzzy number $\widetilde{p} \in \widetilde{\mathcal{P}}'(\mathbb{R})$ with the truncated quadratic membership function

$$\mu_{\widetilde{p}}(x) = \begin{cases} 0 & \text{for} \quad x \leq \overline{x} - \beta_{\mathrm{l}} \\ 1 - (x - \overline{x})^2/\beta_{\mathrm{l}}^2 & \text{for} \quad x - \beta_{\mathrm{l}} < x < \overline{x} \\ 1 - (x - \overline{x})^2/\beta_{\mathrm{r}}^2 & \text{for} \quad \overline{x} \leq x < \overline{x} + \beta_{\mathrm{r}} \\ 0 & \text{for} \quad x \geq \overline{x} + \beta_{\mathrm{r}} \end{cases} \;. \tag{2.15}$$



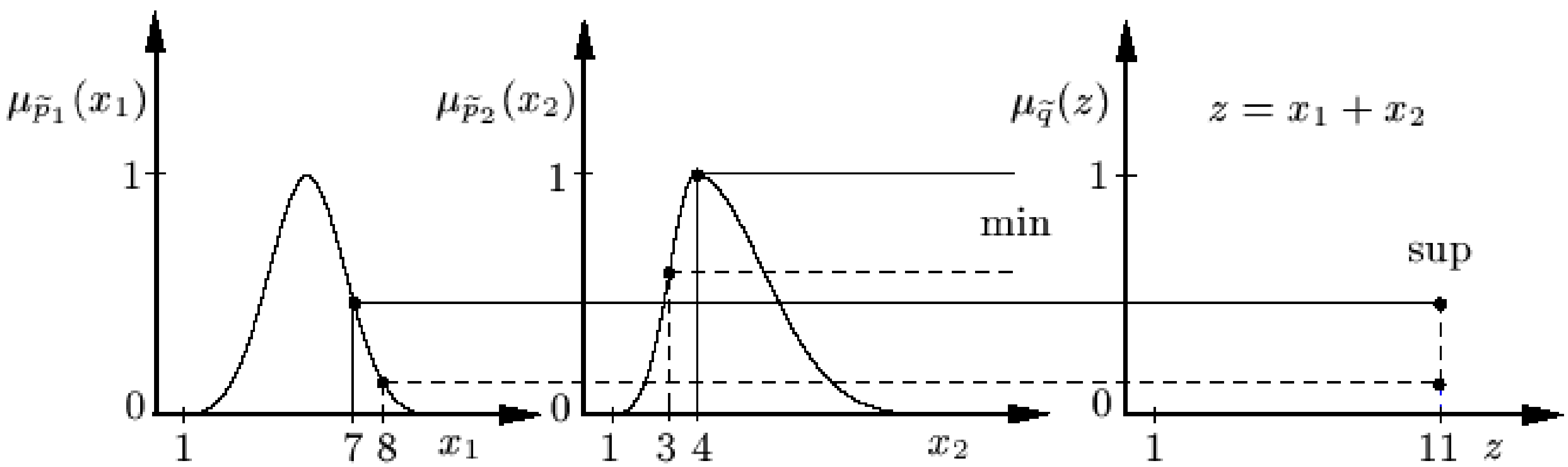


Fig. 2.5. Evaluation of the extension principle for the addition of two fuzzy numbers.

addition

Addition of L-R Fuzzy Numbers

Given two fuzzy numbers $\widetilde{p}_1$ and $\widetilde{p}_2$, represented as L-R fuzzy numbers of the form

$$\widetilde{p}_1 = \langle \overline{x}_1, \alpha_1, \beta_1 \rangle_{L,R} \quad \text{and} \quad \widetilde{p}_2 = \langle \overline{x}_2, \alpha_2, \beta_2 \rangle_{L,R} , \tag{2.37}$$

the sum $E_\mathrm{a}(\widetilde{p}_1, \widetilde{p}_2) = \widetilde{q} = \widetilde{p}_1 + \widetilde{p}_2$ is again an L-R fuzzy number of the form

$$\widetilde{q} = \langle \overline{z}, \alpha, \beta \rangle_{L,R} \tag{2.38}$$

with the modal value

$$\overline{z} = \overline{x}_1 + \overline{x}_2 \tag{2.39}$$

and the spreads

$$\alpha = \alpha_1 + \alpha_2 \quad \text{and} \quad \beta = \beta_1 + \beta_2 . \tag{2.40}$$

$$\underbrace{\langle \overline{x}_1, \alpha_1, \beta_1 \rangle_{L,R}}_{\widetilde{p}_1} + \underbrace{\langle \overline{x}_2, \alpha_2, \beta_2 \rangle_{L,R}}_{\widetilde{p}_2} = \underbrace{\langle \overbrace{\overline{x}_1 + \overline{x}_2}^{\overline{z}}, \overbrace{\alpha_1 + \alpha_2}^{\alpha}, \overbrace{\beta_1 + \beta_2}^{\beta} \rangle_{L,R}}_{\widetilde{q}} . \tag{2.41}$$

Subtraction of L-R Fuzzy Numbers

Making use of the opposite $-\widetilde{p}$ of the L-R fuzzy number $\widetilde{p}$, which is defined as

$$-\widetilde{p} = -\langle \overline{x}, \alpha, \beta \rangle_{L,R} = \langle -\overline{x}, \beta, \alpha \rangle_{R,L} \ , \qquad (2.49)$$

we can deduce the following formula from (2.41) for the subtraction $\widetilde{q} = E_s(\widetilde{p}_1, \widetilde{p}_2) = \widetilde{p}_1 - \widetilde{p}_2$ of L-R fuzzy numbers:

$$\underbrace{\langle \overline{x}_1, \alpha_1, \beta_1 \rangle_{L,R}}_{\widetilde{p}_1} - \underbrace{\langle \overline{x}_2, \alpha_2, \beta_2 \rangle_{R,L}}_{\widetilde{p}_2} = \underbrace{\langle \overbrace{\overline{x}_1 - \overline{x}_2}^{\overline{z}}, \overbrace{\alpha_1 + \beta_2}^{\alpha}, \overbrace{\beta_1 + \alpha_2}^{\beta} \rangle_{L,R}}_{\widetilde{q}} \ . \qquad (2.50)$$

Let us consider again two fuzzy numbers $\widetilde{p}_1$ and $\widetilde{p}_2$ of the same L-R type given by the L-R representations

$$\widetilde{p}_1 = \langle \overline{x}_1, \alpha_1, \beta_1 \rangle_{L,R} \quad \text{and} \quad \widetilde{p}_2 = \langle \overline{x}_2, \alpha_2, \beta_2 \rangle_{L,R} \, . \tag{2.53}$$

$$\underbrace{\langle \overline{x}_1, \alpha_1, \beta_1 \rangle_{L,R}}_{\widetilde{p}_1} \cdot \underbrace{\langle \overline{x}_2, \alpha_2, \beta_2 \rangle_{L,R}}_{\widetilde{p}_2} \approx \underbrace{\langle \overbrace{\overline{x}_1 \overline{x}_2}^{\overline{z}}, \overbrace{\overline{x}_1 \alpha_2 + \overline{x}_2 \alpha_1}^{\alpha}, \overbrace{\overline{x}_1 \beta_2 + \overline{x}_2 \beta_1}^{\beta} \rangle_{L,R}}_{\widetilde{q}_t} \, .$$