Chapter 7

Markov Chains

INTRODUCTION

We review the definitions and elementary properties of vectors and matrices which are required for this chapter.

By a *vector* u we simply mean an n-tuple of numbers:

$$u = (u_1, u_2, \ldots, u_n)$$

The u_i are called the *components* of u. If all the $u_i = 0$, then u is called the *zero vector.* By a *scalar multiple* ku of u (where k is a real number), we mean the vector obtained from u by multiplying its components by k:

$$ku = (ku_1, ku_2, \ldots, ku_n)$$

We note that two vectors are equal if and only if their corresponding components are equal.

By a *matrix* A we mean a rectangular array of numbers:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

The m horizontal n-tuples

$$(a_{11}, a_{12}, \ldots, a_{1n}),\ (a_{21}, a_{22}, \ldots, a_{2n}),\ \ldots,\ (a_{m1}, a_{m2}, \ldots, a_{mn})$$

are called the *rows* of A, and the n vertical m-tuples

$$\begin{pmatrix} a_{11} \\ a_{21} \\ \cdots \\ a_{m1} \end{pmatrix},\quad \begin{pmatrix} a_{12} \\ a_{22} \\ \cdots \\ a_{m2} \end{pmatrix},\quad \ldots,\quad \begin{pmatrix} a_{1n} \\ a_{2n} \\ \cdots \\ a_{mn} \end{pmatrix}$$

its *columns.* Note that the element a_{ij}, called the *ij-entry,* appears in the ith row and the jth column. We frequently denote such a matrix simply by $A = (a_{ij})$.

A matrix with m rows and n columns is said to be an m by n matrix, written $m \times n$ matrix; if $m = n$, then it is called a *square* matrix (or: *n-square* matrix). We also note that a matrix with only one row may be viewed as a vector, and vice versa.

Now suppose A and B are two matrices such that the number of columns of A is equal to the number of rows of B, say A is an $m \times p$ matrix and B is a $p \times n$ matrix. Then the product of A and B, written AB, is the $m \times n$ matrix whose ij-entry is obtained by multiplying the elements of the ith row of A by the corresponding elements of the jth column of B and then adding:

$$\begin{pmatrix} a_{11} & \cdots & a_{1p} \\ \cdot & \cdots & \cdot \\ a_{i1} & \cdots & a_{ip} \\ \cdot & \cdots & \cdot \\ a_{m1} & \cdots & a_{mp} \end{pmatrix} \begin{pmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1n} \\ \cdot & \cdots & \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot & \cdots & \cdot \\ b_{p1} & \cdots & b_{pj} & \cdots & b_{pn} \end{pmatrix} = \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \cdot & \cdots & \cdot \\ \cdot & c_{ij} & \cdot \\ \cdot & \cdots & \cdot \\ c_{m1} & \cdots & c_{mn} \end{pmatrix}$$

where
$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj} = \sum_{k=1}^{p} a_{ik}b_{kj}$$

If the number of columns of A is not equal to the number of rows of B, say A is $m \times p$ and B is $q \times n$ where $p \neq q$, then the product AB is not defined.

There are special cases of matrix multiplication which are of special interest. If A is an n-square matrix, then we can form all the *powers* of A:

$$A^2 = AA, \; A^3 = AA^2, \; A^4 = AA^3, \; \ldots$$

In addition, if u is a vector with n components, then we can form the product

$$uA$$

which is again a vector with n components. We call $u \neq 0$ a *fixed vector* (or: *fixed point*) of A, if u is "left fixed", i.e. is not changed, when multiplied by A:

$$uA = u$$

In this case, for any scalar $k \neq 0$, we have

$$(ku)A = k(uA) = ku$$

That is,

Theorem 7.1: If u is a fixed vector of a matrix A, then every nonzero scalar multiple ku of u is also a fixed vector of A.

Example 7.1:
$$\begin{pmatrix} r & s \\ t & u \end{pmatrix}\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} = \begin{pmatrix} ra_1 + sb_1 & ra_2 + sb_2 & ra_3 + sb_3 \\ ta_1 + ub_1 & ta_2 + ub_2 & ta_3 + ub_3 \end{pmatrix}$$

Example 7.2: If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, then

$$A^2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1+6 & 2+8 \\ 3+12 & 6+16 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$$

Example 7.3:
$$(1, 2, 3)\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = (1+8+21, 2+10+24, 3+12+27) = (30, 36, 42)$$

Example 7.4: Consider the matrix $A = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$. Then the vector $u = (2, -1)$ is a fixed point of A. For,

$$uA = (2, -1)\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} = (2 \cdot 2 - 1 \cdot 2, \; 2 \cdot 1 - 1 \cdot 3) = (2, -1) = u$$

Thus by the above theorem, the vector $2u = (4, -2)$ is also a fixed point of A:

$$(4, -2)\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} = (4 \cdot 2 - 2 \cdot 2, \; 4 \cdot 1 - 2 \cdot 3) = (4, -2)$$

PROBABILITY VECTORS, STOCHASTIC MATRICES

A vector $u = (u_1, u_2, \ldots, u_n)$ is called a *probability vector* if the components are nonnegative and their sum is 1.

Example 7.5: Consider the following vectors:

$$u = (\tfrac{3}{4}, 0, -\tfrac{1}{4}, \tfrac{1}{2}), \quad v = (\tfrac{3}{4}, \tfrac{1}{2}, 0, \tfrac{1}{4}) \quad \text{and} \quad w = (\tfrac{1}{4}, \tfrac{1}{4}, 0, \tfrac{1}{2})$$

Then:

u is not a probability vector since its third component is negative;

v is not a probability vector since the sum of its components is greater than 1;

w is a probability vector.

Example 7.6: The nonzero vector $v = (2,3,5,0,1)$ is not a probability vector since the sum of its components is $2+3+5+0+1 = 11$. However, since the components of v are nonnegative, v has a unique scalar multiple λv which is a probability vector; it can be obtained from v by multiplying each component of v by the reciprocal of the sum of the components of v: $\frac{1}{11}v = (\frac{2}{11}, \frac{3}{11}, \frac{5}{11}, 0, \frac{1}{11})$.

A square matrix $P = (p_{ij})$ is called a *stochastic matrix* if each of its rows is a probability vector, i.e. if each entry of P is nonnegative and the sum of the entries in each row is 1.

Example 7.7: Consider the following matrices:

$$\begin{pmatrix} \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{3}{4} & \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \qquad \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix}$$
$$\text{(i)} \qquad\qquad\qquad \text{(ii)} \qquad\qquad\qquad \text{(iii)}$$

(i) is not a stochastic matrix since the entry in the second row and third column is negative;

(ii) is not a stochastic matrix since the sum of the entries in the second row is not 1;

(iii) is a stochastic matrix since each row is a probability vector.

We shall prove (see Problem 7.10)

Theorem 7.2: If A and B are stochastic matrices, then the product AB is a stochastic matrix. Therefore, in particular, all powers A^n are stochastic matrices.

REGULAR STOCHASTIC MATRICES

We now define an important class of stochastic matrices whose properties shall be investigated subsequently.

Definition: A stochastic matrix P is said to be *regular* if all the entries of some power P^m are positive.

Example 7.8: The stochastic matrix $A = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ is regular since

$$A^2 = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

is positive in every entry.

Example 7.9: Consider the stochastic matrix $A = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$. Here

$$A^2 = \begin{pmatrix} 1 & 0 \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}, \quad A^3 = \begin{pmatrix} 1 & 0 \\ \frac{7}{8} & \frac{1}{8} \end{pmatrix}, \quad A^4 = \begin{pmatrix} 1 & 0 \\ \frac{15}{16} & \frac{1}{16} \end{pmatrix}$$

In fact every power A^m will have 1 and 0 in the first row; hence A is not regular.

FIXED POINTS AND REGULAR STOCHASTIC MATRICES

The fundamental property of regular stochastic matrices is contained in the following theorem whose proof lies beyond the scope of this text.

Theorem 7.3: Let P be a regular stochastic matrix. Then:

(i) P has a unique fixed probability vector t, and the components of t are all positive;

(ii) the sequence $P, P^2, P^3, \ldots$ of powers of P approaches the matrix T whose rows are each the fixed point t;

(iii) if p is any probability vector, then the sequence of vectors $pP, pP^2, pP^3, \ldots$ approaches the fixed point t.

Note: P^n approaches T means that each entry of P^n approaches the corresponding entry of T, and pP^n approaches t means that each component of pP^n approaches the corresponding component of t.

Example 7.10: Consider the regular stochastic matrix $P = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$. We seek a probability vector with two components, which we can denote by $t = (x, 1-x)$, such that $tP = t$:

$$(x, 1-x)\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = (x, 1-x)$$

Multiplying the left side of the above matrix equation, we obtain

$$(\tfrac{1}{2} - \tfrac{1}{2}x, \tfrac{1}{2} + \tfrac{1}{2}x) = (x, 1-x) \quad \text{or} \quad \begin{cases} \frac{1}{2} - \frac{1}{2}x = x \\ \frac{1}{2} + \frac{1}{2}x = 1 - x \end{cases} \quad \text{or} \quad x = \tfrac{1}{3}$$

Thus $t = (\frac{1}{3}, 1 - \frac{1}{3}) = (\frac{1}{3}, \frac{2}{3})$ is the unique fixed probability vector of P. By Theorem 7.3, the sequence $P, P^2, P^3, \ldots$ approaches the matrix T whose rows are each the vector t:

$$T = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} = \begin{pmatrix} .33 & .67 \\ .33 & .67 \end{pmatrix}$$

We exhibit some of the powers of P to indicate the above result:

$$P^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} = \begin{pmatrix} .50 & .50 \\ .25 & .75 \end{pmatrix}; \quad P^3 = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{3}{8} & \frac{5}{8} \end{pmatrix} = \begin{pmatrix} .25 & .75 \\ .37 & .63 \end{pmatrix}$$

$$P^4 = \begin{pmatrix} \frac{3}{8} & \frac{5}{8} \\ \frac{5}{16} & \frac{11}{16} \end{pmatrix} = \begin{pmatrix} .37 & .63 \\ .31 & .69 \end{pmatrix}; \quad P^5 = \begin{pmatrix} \frac{5}{16} & \frac{11}{16} \\ \frac{11}{32} & \frac{21}{32} \end{pmatrix} = \begin{pmatrix} .31 & .69 \\ .34 & .66 \end{pmatrix}$$

Example 7.11: Find the unique fixed probability vector of the regular stochastic matrix

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

Method 1. We seek a probability vector with three components, which we can represent by $t = (x, y, 1-x-y)$, such that $tP = t$:

$$(x, y, 1-x-y)\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} = (x, y, 1-x-y)$$

Multiplying the left side of the above matrix equation and then setting corresponding components equal to each other, we obtain the system

$$\begin{array}{rcl} \tfrac{1}{2} - \tfrac{1}{2}x - \tfrac{1}{2}y &=& x \\ x + \tfrac{1}{2} - \tfrac{1}{2}x - \tfrac{1}{2}y &=& y \\ y &=& 1 - x - y \end{array} \quad \text{or} \quad \begin{array}{rcl} 3x + y &=& 1 \\ x - 3y &=& -1 \\ x + 2y &=& 1 \end{array} \quad \text{or} \quad \begin{array}{rcl} x &=& \tfrac{1}{5} \\ y &=& \tfrac{2}{5} \end{array}$$

Thus $t = (\frac{1}{5}, \frac{2}{5}, \frac{2}{5})$ is the unique fixed probability vector of P.

Method 2. We first seek any fixed vector $u = (x, y, z)$ of the matrix P:

$$(x, y, z)\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} = (x, y, z) \qquad \text{or} \qquad \begin{cases} \frac{1}{2}z = x \\ x + \frac{1}{2}z = y \\ y = z \end{cases}$$

We know that the system has a nonzero solution; hence we can arbitrarily assign a value to one of the unknowns. Set $z = 2$. Then by the first equation $x = 1$, and by the third equation $y = 2$. Thus $u = (1, 2, 2)$ is a fixed point of P. But every multiple of u is a fixed point of P; hence multiply u by $\frac{1}{5}$ to obtain the required fixed probability vector $t = \frac{1}{5}u = (\frac{1}{5}, \frac{2}{5}, \frac{2}{5})$.

MARKOV CHAINS

We now consider a sequence of trials whose outcomes, say, $X_1, X_2, \ldots,$ satisfy the following two properties:

(i) Each outcome belongs to a finite set of outcomes $\{a_1, a_2, \ldots, a_m\}$ called the *state space* of the system; if the outcome on the nth trial is a_i, then we say that the system is in state a_i at time n or at the nth step.

(ii) The outcome of any trial depends at most upon the outcome of the immediately preceding trial and not upon any other previous outcome; with each pair of states (a_i, a_j) there is given the probability p_{ij} that a_j occurs immediately after a_i occurs.

Such a stochastic process is called a (finite) *Markov chain*. The numbers p_{ij}, called the *transition probabilities*, can be arranged in a matrix

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{pmatrix}$$

called the *transition matrix*.

Thus with each state a_i there corresponds the ith row $(p_{i1}, p_{i2}, \ldots, p_{im})$ of the transition matrix P; if the system is in state a_i, then this row vector represents the probabilities of all the possible outcomes of the next trial and so it is a probability vector. Accordingly,

Theorem 7.4: The transition matrix P of a Markov chain is a stochastic matrix.

Example 7.12: A man either drives his car or takes a train to work each day. Suppose he never takes the train two days in a row; but if he drives to work, then the next day he is just as likely to drive again as he is to take the train.

The state space of the system is $\{t\,(\text{train}),\ d\,(\text{drive})\}$. This stochastic process is a Markov chain since the outcome on any day depends only on what happened the preceding day. The transition matrix of the Markov chain is

$$\begin{array}{c c} & \begin{array}{cc} t & d \end{array} \\ \begin{array}{c} t \\ d \end{array} & \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \end{array}$$

The first row of the matrix corresponds to the fact that he never takes the train two days in a row and so he definitely will drive the day after he takes the train. The second row of the matrix corresponds to the fact that the day after he drives he will drive or take the train with equal probability.

Example 7.13: Three boys A, B and C are throwing a ball to each other. A always throws the ball to B and B always throws the ball to C; but C is just as likely to throw the ball to B as to A. Let X_n denote the nth person to be thrown the ball. The state space of the system is $\{A, B, C\}$. This is a Markov chain since the person throwing the ball is not influenced by those who previously had the ball. The transition matrix of the Markov chain is

$$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \end{array}$$

The first row of the matrix corresponds to the fact that A always throws the ball to B. The second row corresponds to the fact that B always throws the ball to C. The last row corresponds to the fact that C throws the ball to A or B with equal probability (and does not throw it to himself).

Example 7.14: A school contains 200 boys and 150 girls. One student is selected after another to take an eye examination. Let X_n denote the sex of the nth student who takes the examination. The state space of the stochastic process is $\{m\,(\text{male}), f\,(\text{female})\}$. However, this process is not a Markov chain since, for example, the probability that the third person is a girl depends not only on the outcome of the second trial but on both the first and second trials.

Example 7.15: (Random walk with reflecting barriers.) A man is at an integral point on the x-axis between the origin O and, say, the point 5. He takes a unit step to the right with probability p or to the left with probability $q = 1 - p$, unless he is at the origin where he takes a step to the right to 1 or at the point 5 where he takes a step to the left to 4. Let X_n denote his position after n steps. This is a Markov chain with state space $\{a_0, a_1, a_2, a_3, a_4, a_5\}$ where a_i means that the man is at the point i. The transition matrix is

$$P \quad = \quad \begin{array}{c} \\ a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{array}\begin{array}{c} \begin{array}{cccccc} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 & 0 \\ 0 & q & 0 & p & 0 & 0 \\ 0 & 0 & q & 0 & p & 0 \\ 0 & 0 & 0 & q & 0 & p \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Each row of the matrix, except the first and last, corresponds to the fact that the man moves from state a_i to state a_{i+1} with probability p or back to state a_{i-1} with probability $q = 1 - p$. The first row corresponds to the fact that the man must move from state a_0 to state a_1, and the last row that the man must move from state a_5 to state a_4.

HIGHER TRANSITION PROBABILITIES

The entry p_{ij} in the transition matrix P of a Markov chain is the probability that the system changes from the state a_i to the state a_j in one step: $a_i \to a_j$. Question: What is the probability, denoted by $p_{ij}^{(n)}$, that the system changes from the state a_i to the state a_j in exactly n steps:

$$a_i \to a_{k_1} \to a_{k_2} \to \cdots \to a_{k_{n-1}} \to a_j$$

The next theorem answers this question; here the $p_{ij}^{(n)}$ are arranged in a matrix $P^{(n)}$ called the *n-step transition matrix*:

Theorem 7.5: Let P be the transition matrix of a Markov chain process. Then the n-step transition matrix is equal to the nth power of P; that is, $P^{(n)} = P^n$.

Now suppose that, at some arbitrary time, the probability that the system is in state a_i is p_i; we denote these probabilities by the probability vector $p = (p_1, p_2, \ldots, p_m)$ which is called the *probability distribution* of the system at that time. In particular, we shall let

$$p^{(0)} = (p_1^{(0)}, p_2^{(0)}, \ldots, p_m^{(0)})$$

denote the *initial probability distribution,* i.e. the distribution when the process begins, and we shall let

$$p^{(n)} = (p_1^{(n)}, p_2^{(n)}, \ldots, p_m^{(n)})$$

denote the nth *step probability distribution*, i.e. the distribution after the first n steps. The following theorem applies.

Theorem 7.6: Let P be the transition matrix of a Markov chain process. If $p = (p_i)$ is the probability distribution of the system at some arbitrary time, then pP is the probability distribution of the system one step later and pP^n is the probability distribution of the system n steps later. In particular,

$$p^{(1)} = p^{(0)}P, \quad p^{(2)} = p^{(1)}P, \quad p^{(3)} = p^{(2)}P, \quad \ldots, \quad p^{(n)} = p^{(0)}P^n$$

Example 7.16: Consider the Markov chain of Example 7.12 whose transition matrix is

$$P = \begin{array}{c} \\ t \\ d \end{array} \begin{array}{c} \begin{array}{cc} t & d \end{array} \\ \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \end{array}$$

Here t is the state of taking a train to work and d of driving to work. By Example 7.8,

$$P^4 = P^2 \cdot P^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} = \begin{pmatrix} \frac{3}{8} & \frac{5}{8} \\ \frac{5}{16} & \frac{11}{16} \end{pmatrix}$$

Thus the probability that the system changes from, say, state t to state d in exactly 4 steps is $\frac{5}{8}$, i.e. $p_{td}^{(4)} = \frac{5}{8}$. Similarly, $p_{tt}^{(4)} = \frac{3}{8}$, $p_{dt}^{(4)} = \frac{5}{16}$ and $p_{dd}^{(4)} = \frac{11}{16}$.

Now suppose that on the first day of work, the man tossed a fair die and drove to work if and only if a 6 appeared. In other words, $p^{(0)} = (\frac{5}{6}, \frac{1}{6})$ is the initial probability distribution. Then

$$p^{(4)} = p^{(0)}P^4 = (\tfrac{5}{6}, \tfrac{1}{6})\begin{pmatrix} \frac{3}{8} & \frac{5}{8} \\ \frac{5}{16} & \frac{11}{16} \end{pmatrix} = (\tfrac{35}{96}, \tfrac{61}{96})$$

is the probability distribution after 4 days, i.e. $p_t^{(4)} = \frac{35}{96}$ and $p_d^{(4)} = \frac{61}{96}$.

Example 7.17: Consider the Markov chain of Example 7.13 whose transition matrix is

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \end{array}$$

Suppose C was the first person with the ball, i.e. suppose $p^{(0)} = (0, 0, 1)$ is the initial probability distribution. Then

$$p^{(1)} = p^{(0)}P = (0, 0, 1)\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} = (\tfrac{1}{2}, \tfrac{1}{2}, 0)$$

$$p^{(2)} = p^{(1)}P = (\tfrac{1}{2}, \tfrac{1}{2}, 0)\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} = (0, \tfrac{1}{2}, \tfrac{1}{2})$$

$$p^{(3)} = p^{(2)}P = (0, \tfrac{1}{2}, \tfrac{1}{2})\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} = (\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{2})$$

Thus, after three throws, the probability that A has the ball is $\frac{1}{4}$, that B has the ball is $\frac{1}{4}$ and that C has the ball is $\frac{1}{2}$: $p_A^{(3)} = \frac{1}{4}$, $p_B^{(3)} = \frac{1}{4}$ and $p_C^{(3)} = \frac{1}{2}$.

Example 7.18: Consider the random walk problem of Example 7.15. Suppose the man began at the point 2; find the probability distribution after 3 steps and after 4 steps, i.e. $p^{(3)}$ and $p^{(4)}$.

Now $p^{(0)} = (0, 0, 1, 0, 0, 0)$ is the initial probability distribution. Then

$$p^{(1)} = p^{(0)}P = (0, q, 0, p, 0, 0)$$

$$p^{(2)} = p^{(1)}P = (q^2, 0, 2pq, 0, p^2, 0)$$

$$p^{(3)} = p^{(2)}P = (0, q^2 + 2pq^2, 0, 3p^2q, 0, p^3)$$

$$p^{(4)} = p^{(3)}P = (q^3 + 2pq^3, 0, pq^2 + 5p^2q^2, 0, 3p^3q + p^3, 0)$$

Thus after 4 steps he is at, say, the origin with probability $q^3 + 2pq^3$.

STATIONARY DISTRIBUTION OF REGULAR MARKOV CHAINS

Suppose that a Markov chain is regular, i.e. that its transition matrix P is regular. By Theorem 7.3 the sequence of n-step transition matrices P^n approaches the matrix T whose rows are each the unique fixed probability vector t of P; hence the probability $p_{ij}^{(n)}$ that a_j occurs for sufficiently large n is independent of the original state a_i and it approaches the component t_j of t. In other words,

Theorem 7.7: Let the transition matrix P of a Markov chain be regular. Then, in the long run, the probability that any state a_j occurs is approximately equal to the component t_j of the unique fixed probability vector t of P.

Thus we see that the effect of the initial state or the initial probability distribution of the process wears off as the number of steps of the process increase. Furthermore, every sequence of probability distributions approaches the fixed probability vector t of P, called the *stationary distribution* of the Markov chain.

Example 7.19: Consider the Markov chain process of Example 7.12 whose transition matrix is

$$P = \begin{array}{c c} & \begin{array}{cc} t & d \end{array} \\ \begin{array}{c} t \\ d \end{array} & \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \end{array}$$

By Example 7.10, the unique fixed probability vector of the above matrix is $(\frac{1}{3}, \frac{2}{3})$. Thus, in the long run, the man will take the train to work $\frac{1}{3}$ of the time, and drive to work the other $\frac{2}{3}$ of the time.

Example 7.20: Consider the Markov chain process of Example 7.13 whose transition matrix is

$$P \;=\; \begin{array}{c} \\ A \\ B \\ C \end{array} \!\!\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \end{array}$$

By Example 7.11, the unique fixed probability vector of the above matrix is $(\frac{1}{5}, \frac{2}{5}, \frac{2}{5})$. Thus, in the long run, A will be thrown the ball 20% of the time, and B and C 40% of the time.

ABSORBING STATES

A state a_i of a Markov chain is called *absorbing* if the system remains in the state a_i once it enters there. Thus a state a_i is absorbing if and only if the ith row of the transition matrix P has a 1 on the main diagonal and zeros everywhere else. (The *main diagonal* of an n-square matrix $A = (a_{ij})$ consists of the entries $a_{11}, a_{22}, \ldots, a_{nn}$.)

Example 7.21: Suppose the following matrix is the transition matrix of a Markov chain:

$$P \;=\; \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{array} \!\!\begin{array}{c} \begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & a_5 \end{array} \\ \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

The states a_2 and a_5 are each absorbing, since each of the second and fifth rows has a 1 on the main diagonal.

Example 7.22: (Random walk with absorbing barriers.) Consider the random walk problem of Example 7.15, except now we assume that the man remains at either endpoint whenever he reaches there. This is also a Markov chain and the transition matrix is given by

$$P \;=\; \begin{array}{c} \\ a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{array} \!\!\begin{array}{c} \begin{array}{cccccc} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 & 0 \\ 0 & q & 0 & p & 0 & 0 \\ 0 & 0 & q & 0 & p & 0 \\ 0 & 0 & 0 & q & 0 & p \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

We call this process a random walk with absorbing barriers, since the a_0 and a_5 are absorbing states. In this case, $p_0^{(n)}$ denotes the probability that the man reaches the state a_0 on or before the nth step. Similarly, $p_5^{(n)}$ denotes the probability that he reaches the state a_5 on or before the nth step.

Example 7.23: A player has, say, x dollars. He bets one dollar at a time and wins with probability p and loses with probability $q = 1 - p$. The game ends when he loses all his money, i.e. has 0 dollars, or when he wins $N - x$ dollars, i.e. has N dollars. This game is identical to the random walk of the preceding example except that here the absorbing barriers are at 0 and N.

Example 7.24: A man tosses a fair coin until 3 heads occur in a row. Let $X_n = k$ if, at the nth trial, the last tail occurred at the $(n-k)$-th trial, i.e. X_n denotes the longest string of heads ending at the nth trial. This is a Markov chain process with state space $\{a_0, a_1, a_2, a_3\}$, where a_i means the string of heads has length i. The transition matrix is

$$\begin{array}{c} \\ a_0 \\ a_1 \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{cccc} a_0 & a_1 & a_2 & a_3 \end{array} \\ \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Each row, except the last, corresponds to the fact that a string of heads is either broken if a tail occurs or is extended by one if a head occurs. The last line corresponds to the fact that the game ends if three heads are tossed in a row. Note that a_3 is an absorbing state.

Let a_i be an absorbing state of a Markov chain with transition matrix P. Then, for $j \neq i$, the n-step transition probability $p_{ij}^{(n)} = 0$ for every n. Accordingly, every power of P has a zero entry and so P is not regular. Thus:

Theorem 7.8: If a stochastic matrix P has a 1 on the main diagonal, then P is not regular (unless P is a 1×1 matrix).

Solved Problems

MATRIX MULTIPLICATION

7.1. Let $u = (1, -2, 4)$ and $A = \begin{pmatrix} 1 & 3 & -1 \\ 0 & 2 & 5 \\ 4 & 1 & 6 \end{pmatrix}$. Find uA.

The product of the vector u with 3 components by the 3×3 matrix A is again a vector with 3 components. To obtain the first component of uA, multiply the elements of u by the corresponding elements of the first column of A and then add:

$$(1, -2, 4)\begin{pmatrix} 1 & 3 & -1 \\ 0 & 2 & 5 \\ 4 & 1 & 6 \end{pmatrix} = (1 \cdot 1 + (-2) \cdot 0 + 4 \cdot 4, \quad , \quad) = (17, \ , \ )$$

To obtain the second component of uA, multiply the elements of u by the corresponding elements of the second column of A and then add:

$$(1, -2, 4)\begin{pmatrix} 1 & 3 & -1 \\ 0 & 2 & 5 \\ 4 & 1 & 6 \end{pmatrix} = (17, 1 \cdot 3 + (-2) \cdot 2 + 4 \cdot 1, \quad) = (17, 3, \ )$$

To obtain the third component of uA, multiply the elements of u by the corresponding elements of the third column of A and then add:

$$(1, -2, 4)\begin{pmatrix} 1 & 3 & -1 \\ 0 & 2 & 5 \\ 4 & 1 & 6 \end{pmatrix} = (17, 3, 1 \cdot (-1) + (-2) \cdot 5 + 4 \cdot 6) = (17, 3, 13)$$

That is,

$$uA = (17, 3, 13)$$

7.2. Let $A = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 & -4 \\ 3 & -2 & 6 \end{pmatrix}$. Find (i) AB, (ii) BA.

(i) Since A is 2×2 and B is 2×3, the product AB is a 2×3 matrix. To obtain the first row of AB, multiply the elements of the first row $(1, 3)$ of A by the corresponding elements of each of the columns $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 6 \end{pmatrix}$ of B and then add:

$$\begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 & -4 \\ 3 & -2 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 1\cdot 2 + 3\cdot 3 & 1\cdot 0 + 3\cdot(-2) & 1\cdot(-4) + 3\cdot 6 \\ & & \end{pmatrix} = \begin{pmatrix} 11 & -6 & 14 \\ & & \end{pmatrix}$$

To obtain the second row of AB, multiply the elements of the second row $(2, -1)$ of A by the corresponding elements of each of the columns of B and then add:

$$\begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 & -4 \\ 3 & -2 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & -6 & 14 \\ 2\cdot 2 + (-1)\cdot 3 & 2\cdot 0 + (-1)\cdot(-2) & 2\cdot(-4) + (-1)\cdot 6 \end{pmatrix} = \begin{pmatrix} 11 & -6 & 14 \\ 1 & 2 & -14 \end{pmatrix}$$

Thus $$AB = \begin{pmatrix} 11 & -6 & 14 \\ 1 & 2 & -14 \end{pmatrix}$$

(ii) Note B is 2×3 and A is 2×2. Since the "inner numbers" 3 and 2 are not equal, i.e. the number of columns of B is not equal to the number of rows of A, the product BA is not defined.

7.3. Let $A = \begin{pmatrix} 1 & 2 \\ 4 & -3 \end{pmatrix}$. Find (i) A^2, (ii) A^3.

(i) $$A^2 = AA = \begin{pmatrix} 1 & 2 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 4 & -3 \end{pmatrix}$$
$$= \begin{pmatrix} 1\cdot 1 + 2\cdot 4 & 1\cdot 2 + 2\cdot(-3) \\ 4\cdot 1 + (-3)\cdot 4 & 4\cdot 2 + (-3)\cdot(-3) \end{pmatrix} = \begin{pmatrix} 9 & -4 \\ -8 & 17 \end{pmatrix}$$

(ii) $$A^3 = AA^2 = \begin{pmatrix} 1 & 2 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} 9 & -4 \\ -8 & 17 \end{pmatrix}$$
$$= \begin{pmatrix} 1\cdot 9 + 2\cdot(-8) & 1\cdot(-4) + 2\cdot 17 \\ 4\cdot 9 + (-3)\cdot(-8) & 4\cdot(-4) + (-3)\cdot 17 \end{pmatrix} = \begin{pmatrix} -7 & 30 \\ 60 & -67 \end{pmatrix}$$

PROBABILITY VECTORS AND STOCHASTIC MATRICES

7.4. Which vectors are probability vectors?

(i) $u = (\frac{1}{3}, 0, -\frac{1}{6}, \frac{1}{2}, \frac{1}{3})$, (ii) $v = (\frac{1}{3}, 0, \frac{1}{6}, \frac{1}{2}, \frac{1}{3})$, (iii) $w = (\frac{1}{3}, 0, 0, \frac{1}{6}, \frac{1}{2})$.

A vector is a probability vector if its components are nonnegative and their sum is 1.

(i) u is not a probability vector since its third component is negative.

(ii) v is not a probability vector since the sum of the components is greater than 1.

(iii) w is a probability vector since the components are nonnegative and their sum is 1.

7.5. Multiply each vector by the appropriate scalar to form a probability vector:

(i) $(2, 1, 0, 2, 3)$, (ii) $(4, 0, 1, 2, 0, 5)$, (iii) $(3, 0, -2, 1)$, (iv) $(0, 0, 0, 0, 0)$.

(i) The sum of the components is $2 + 1 + 0 + 3 + 2 = 8$; hence multiply the vector, i.e. each component, by $\frac{1}{8}$ to obtain the probability vector $(\frac{1}{4}, \frac{1}{8}, 0, \frac{1}{4}, \frac{3}{8})$.

(ii) The sum of the components is $4+0+1+2+0+5 = 12$; hence multiply the vector, i.e. each component, by $\frac{1}{12}$ to obtain the probability vector $(\frac{1}{3}, 0, \frac{1}{12}, \frac{1}{6}, 0, \frac{5}{12})$.

(iii) The first component is positive and the third is negative; hence it is impossible to multiply the vector by a scalar to form a vector with nonnegative components. Thus no scalar multiple of the vector is a probability vector.

(iv) Every scalar multiple of the zero vector is the zero vector whose components add to 0. Thus no multiple of the zero vector is a probability vector.

7.6. Find a multiple of each vector which is a probability vector:
(i) $(\frac{1}{2}, \frac{2}{3}, 0, 2, \frac{5}{6})$, (ii) $(0, \frac{2}{3}, 1, \frac{3}{5}, \frac{5}{6})$.

In each case, first multiply each vector by a scalar so that the fractions are eliminated.

(i) First multiply the vector by 6 to obtain $(3, 4, 0, 12, 5)$. Then multiply by $1/(3+4+0+12+5) = \frac{1}{24}$ to obtain $(\frac{1}{8}, \frac{1}{6}, 0, \frac{1}{2}, \frac{5}{24})$ which is a probability vector.

(ii) First multiply the vector by 30 to obtain $(0, 20, 30, 18, 25)$. Then multiply by $1/(0+20+30+18+25) = \frac{1}{93}$ to obtain $(0, \frac{20}{93}, \frac{30}{93}, \frac{18}{93}, \frac{25}{93})$ which is a probability vector.

7.7. Which of the following matrices are stochastic matrices?

(i) $A = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$ (ii) $B = \begin{pmatrix} \frac{15}{16} & \frac{1}{16} \\ \frac{2}{3} & \frac{2}{3} \end{pmatrix}$ (iii) $C = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ (iv) $D = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$.

(i) A is not a stochastic matrix since it is not a square matrix.

(ii) B is not a stochastic matrix since the sum of the components in the last row is greater than 1.

(iii) C is a stochastic matrix.

(iv) D is not a stochastic matrix since the entry in the first row, second column is negative.

7.8. Let $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ be a stochastic matrix and let $u = (u_1, u_2, u_3)$ be a probability vector. Show that uA is also a probability vector.

$$uA = (u_1, u_2, u_3) \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = (u_1a_1 + u_2a_2 + u_3a_3,\ u_1b_1 + u_2b_2 + u_3b_3,\ u_1c_1 + u_2c_2 + u_3c_3)$$

Since the u_i, a_i, b_i and c_i are nonnegative and since the products and sums of nonnegative numbers are nonnegative, the components of uA are nonnegative as required. Thus we only need to show that the sum of the components of uA is 1. Here we use the fact that $u_1 + u_2 + u_3$, $a_1 + b_1 + c_1$, $a_2 + b_2 + c_2$ and $a_3 + b_3 + c_3$ are each 1:

$$\begin{aligned} & u_1a_1 + u_2a_2 + u_3a_3 + u_1b_1 + u_2b_2 + u_3b_3 + u_1c_1 + u_2c_2 + u_3c_3 \\ &= u_1(a_1 + b_1 + c_1) + u_2(a_2 + b_2 + c_2) + u_3(a_3 + b_3 + c_3) \\ &= u_1 \cdot 1 + u_2 \cdot 1 + u_3 \cdot 1 = u_1 + u_2 + u_3 = 1 \end{aligned}$$

7.9. Prove: If $A = (a_{ij})$ is a stochastic matrix of order n and $u = (u_1, u_2, \ldots, u_n)$ is a probability vector, then uA is also a probability vector.

The proof is similar to that of the preceding problem for the case $n = 3$:

$$uA = (u_1, u_2, \ldots, u_n)\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$= (u_1a_{11} + u_2a_{21} + \cdots + u_na_{n1},\ u_1a_{12} + u_2a_{22} + \cdots + u_na_{n2},\ \ldots,\ u_1a_{1n} + u_2a_{2n} + \cdots + u_na_{nn})$$

Since the u_i and a_{ij} are nonnegative, the components of uA are also nonnegative. Thus we only need to show that the sum of the components of uA is 1:

$$u_1a_{11} + u_2a_{21} + \cdots + u_na_{n1} + u_1a_{12} + u_2a_{22} + \cdots + u_na_{n2} + \cdots + u_1a_{1n} + u_2a_{2n} + \cdots + u_na_{nn}$$
$$= u_1(a_{11} + a_{12} + \cdots + a_{1n}) + u_2(a_{21} + a_{22} + \cdots + a_{2n}) + \cdots + u_n(a_{n1} + a_{n2} + \cdots + a_{nn})$$
$$= u_1 \cdot 1 + u_2 \cdot 1 + \cdots + u_n \cdot 1 = u_1 + u_2 + \cdots + u_n = 1$$

7.10. Prove Theorem 7.2: If A and B are stochastic matrices, then the product AB is a stochastic matrix. Therefore, in particular, all powers A^n are stochastic matrices.

The ith-row s_i of the product matrix AB is obtained by multiplying the ith-row r_i of A by the matrix B: $s_i = r_iB$. Since each r_i is a probability vector and B is a stochastic matrix, by the preceding problem, s_i is also a probability vector. Hence AB is a stochastic matrix.

7.11. Prove: Let $p = (p_1, p_2, \ldots, p_m)$ be a probability vector, and let T be a matrix whose rows are each the same vector $t = (t_1, t_2, \ldots, t_m)$. Then $pT = t$.

Using the fact that $p_1 + p_2 + \cdots + p_m = 1$, we have

$$pT = (p_1, p_2, \ldots, p_m)\begin{pmatrix} t_1 & t_2 & \cdots & t_m \\ t_1 & t_2 & \cdots & t_m \\ \cdots & \cdots & \cdots & \cdots \\ t_1 & t_2 & \cdots & t_m \end{pmatrix}$$
$$= (p_1t_1 + p_2t_1 + \cdots + p_mt_1,\ p_1t_2 + p_2t_2 + \cdots + p_mt_2,\ \ldots,\ p_1t_m + p_2t_m + \cdots + p_mt_m)$$
$$= ((p_1 + p_2 + \cdots + p_m)t_1,\ (p_1 + p_2 + \cdots + p_m)t_2,\ \ldots,\ (p_1 + p_2 + \cdots + p_m)t_m)$$
$$= (1 \cdot t_1, 1 \cdot t_2, \ldots, 1 \cdot t_m) = (t_1, t_2, \ldots, t_m) = t$$

REGULAR STOCHASTIC MATRICES AND FIXED PROBABILITY VECTORS

7.12. Find the unique fixed probability vector of the regular stochastic matrix $A = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$. What matrix does A^n approach?

We seek a probability vector $t = (x, 1-x)$ such that $tA = t$:

$$(x, 1-x)\begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = (x, 1-x)$$

Multiply the left side of the above matrix equation and then set corresponding components equal to each other to obtain the two equations

$$\tfrac{3}{4}x + \tfrac{1}{2} - \tfrac{1}{2}x = x, \qquad \tfrac{1}{4}x + \tfrac{1}{2} - \tfrac{1}{2}x = 1 - x$$

Solve either equation to obtain $x = \frac{2}{3}$. Thus $t = (\frac{2}{3}, \frac{1}{3})$ is the required probability vector.

Check the answer by computing the product tA:

$$(\tfrac{2}{3}, \tfrac{1}{3})\begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = (\tfrac{1}{2} + \tfrac{1}{6}, \tfrac{1}{6} + \tfrac{1}{6}) = (\tfrac{2}{3}, \tfrac{1}{3})$$

The answer checks since $tA = t$.

The matrix A^n approaches the matrix T whose rows are each the fixed point t: $T = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$.

7.13. (i) Show that the vector $u = (b, a)$ is a fixed point of the general 2×2 stochastic matrix $P = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$.

(ii) Use the result of (i) to find the unique fixed probability vector of each of the following matrices:

$$A = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix} \qquad C = \begin{pmatrix} .7 & .3 \\ .8 & .2 \end{pmatrix}$$

(i) $$uP = (b, a)\begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix} = (b - ab + ab,\ ab + a - ab) = (b, a) = u.$$

(ii) By (i), $u = (1, \frac{2}{3})$ is a fixed point of A. Multiply u by 3 to obtain the fixed point $(3, 2)$ of A which has no fractions. Then multiply $(3, 2)$ by $1/(3+2) = \frac{1}{5}$ to obtain the required unique fixed probability vector $(\frac{3}{5}, \frac{2}{5})$.

By (i), $u = (\frac{2}{3}, \frac{1}{2})$ is a fixed point of B. Multiply u by 6 to obtain the fixed point $(4, 3)$, and then multiply by $1/(4+3) = \frac{1}{7}$ to obtain the required unique fixed probability vector $(\frac{4}{7}, \frac{3}{7})$.

By (i), $u = (.8, .3)$ is a fixed point of C. Hence $(8, 3)$ and the probability vector $(\frac{8}{11}, \frac{3}{11})$ are also fixed points of C.

7.14. Find the unique fixed probability vector of the regular stochastic matrix

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix}$$

Method 1. We seek a probability vector $t = (x, y, 1-x-y)$ such that $tP = t$:

$$(x, y, 1-x-y)\begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix} = (x, y, 1-x-y)$$

Multiply the left side of the above matrix equation and then set corresponding components equal to each other to obtain the system of three equations

$$\begin{cases} \frac{1}{2}x + \frac{1}{2}y = x \\ \frac{1}{4}x + 1 - x - y = y \\ \frac{1}{4}x + \frac{1}{2}y = 1 - x - y \end{cases} \qquad \text{or} \qquad \begin{cases} x - y = 0 \\ 3x + 8y = 4 \\ 5x + 6y = 4 \end{cases}$$

Choose any two of the equations and solve for x and y to obtain $x = \frac{4}{11}$ and $y = \frac{4}{11}$. Check the solution by substituting for x and y into the third equation. Since $1 - x - y = \frac{3}{11}$, the required fixed probability vector is $t = (\frac{4}{11}, \frac{4}{11}, \frac{3}{11})$.

Method 2. We seek any fixed vector $u = (x, y, z)$ of the matrix P:

$$(x, y, z)\begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix} = (x, y, z)$$

Multiply the left side of the above matrix equation and set corresponding components equal to each other to obtain the system of three equations

$$\begin{cases} \frac{1}{2}x + \frac{1}{2}y = x \\ \frac{1}{4}x + z = y \\ \frac{1}{4}x + \frac{1}{2}y = z \end{cases} \qquad \text{or} \qquad \begin{cases} x - y = 0 \\ x - 4y + 4z = 0 \\ x + 2y - 4z = 0 \end{cases}$$

We know that the system has a nonzero solution; hence we can arbitrarily assign a value to one of the unknowns. Set $y = 4$. Then by the first equation $x = 4$, and by the third equation $z = 3$. Thus $u = (4, 4, 3)$ is a fixed point of P. Multiply u by $1/(4+4+3) = \frac{1}{11}$ to obtain $t = \frac{1}{11}u = (\frac{4}{11}, \frac{4}{11}, \frac{3}{11})$ which is a probability vector and is also a fixed point of P.

7.15. Find the unique fixed probability vector of the regular stochastic matrix

$$P = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

What matrix does P^n approach?

We first seek any fixed vector $u = (x, y, z)$ of the matrix P:

$$(x, y, z)\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix} = (x, y, z)$$

Multiply the left side of the above matrix equation and set corresponding components equal to each other to obtain the system of three equations

$$\begin{cases} \frac{1}{6}y = x \\ x + \frac{1}{2}y + \frac{2}{3}z = y \\ \frac{1}{3}y + \frac{1}{3}z = z \end{cases} \quad \text{or} \quad \begin{cases} y = 6x \\ 6x + 3y + 4z = 6y \\ y + z = 3z \end{cases} \quad \text{or} \quad \begin{cases} y = 6x \\ 6x + 4z = 3y \\ y = 2z \end{cases}$$

We know that the system has a nonzero solution; hence we can arbitrarily assign a value to one of the unknowns. Set $x = 1$. Then by the first equation $y = 6$, and by the last equation $z = 3$. Thus $u = (1, 6, 3)$ is a fixed point of P. Since $1 + 6 + 3 = 10$, the vector $t = (\frac{1}{10}, \frac{6}{10}, \frac{3}{10})$ is the required unique fixed probability vector of P.

P^n approaches the matrix T whose rows are each the fixed point t: $T = \begin{pmatrix} \frac{1}{10} & \frac{6}{10} & \frac{3}{10} \\ \frac{1}{10} & \frac{6}{10} & \frac{3}{10} \\ \frac{1}{10} & \frac{6}{10} & \frac{3}{10} \end{pmatrix}$.

7.16. If $t = (\frac{1}{4}, 0, \frac{1}{2}, \frac{1}{4}, 0)$ is a fixed point of a stochastic matrix P, why is P not regular?

If P is regular then, by Theorem 7.3, P has a unique fixed probability vector, and the components of the vector are positive. Since the components of the given fixed probability vector are not all positive, P cannot be regular.

7.17. Which of the following stochastic matrices are regular?

(i) $A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$ (ii) $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (iii) $C = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$ (iv) $D = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 \end{pmatrix}$

Recall that a stochastic matrix is regular if a power of the matrix has only positive entries.

(i) A is not regular since there is a 1 on the main diagonal (in the second row).

(ii) $B^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ = the identity matrix I

$B^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = B$

Thus every even power of B is the identity matrix I and every odd power of B is the matrix B. Accordingly every power of B has zero entries, and so B is not regular.

(iii) C is not regular since it has a 1 on the main diagonal.

(iv) $$D^2 = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{8} & \frac{5}{16} & \frac{9}{16} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} \quad \text{and} \quad D^3 = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{5}{32} & \frac{41}{64} & \frac{13}{64} \\ \frac{1}{8} & \frac{5}{16} & \frac{9}{16} \end{pmatrix}$$

Since all the entries of D^3 are positive, D is regular.

MARKOV CHAINS

7.18. A student's study habits are as follows. If he studies one night, he is 70% sure not to study the next night. On the other hand, if he does not study one night, he is 60% sure not to study the next night as well. In the long run, how often does he study?

The states of the system are S (studying) and T (not studying). The transition matrix is

$$P = \begin{array}{c} \\ S \\ T \end{array} \begin{array}{c} \begin{array}{cc} S & T \end{array} \\ \begin{pmatrix} .3 & .7 \\ .4 & .6 \end{pmatrix} \end{array}$$

To discover what happens in the long run, we must find the unique fixed probability vector t of P. By Problem 7.13, $u = (.4, .7)$ is a fixed point of P and so $t = (\frac{4}{11}, \frac{7}{11})$ is the required probability vector. Thus in the long run the student studies $\frac{4}{11}$ of the time.

7.19. A psychologist makes the following assumptions concerning the behavior of mice subjected to a particular feeding schedule. For any particular trial 80% of the mice that went right on the previous experiment will go right on this trial, and 60% of those mice that went left on the previous experiment will go right on this trial. If 50% went right on the first trial, what would he predict for (i) the second trial, (ii) the third trial, (iii) the thousandth trial?

The states of the system are R (right) and L (left). The transition matrix is

$$P = \begin{array}{c} \\ R \\ L \end{array} \begin{array}{c} \begin{array}{cc} R & L \end{array} \\ \begin{pmatrix} .8 & .2 \\ .6 & .4 \end{pmatrix} \end{array}$$

The probability distribution for the first trial is $p = (.5, .5)$. To compute the probability distribution for the next step, i.e. the second trial, multiply p by the transition matrix P:

$$(.5, .5)\begin{pmatrix} .8 & .2 \\ .6 & .4 \end{pmatrix} = (.7, .3)$$

Thus on the second trial he predicts that 70% of the mice will go right and 30% will go left. To compute the probability distribution for the third trial, multiply that of the second trial by P:

$$(.7, .3)\begin{pmatrix} .8 & .2 \\ .6 & .4 \end{pmatrix} = (.74, .26)$$

Thus on the third trial he predicts that 74% of the mice will go right and 26% will go left.

We assume that the probability distribution for the thousandth trial is essentially the stationary probability distribution of the Markov chain, i.e. the unique fixed probability vector t of the transition matrix P. By Problem 7.13, $u = (.6, .2)$ is a fixed point of P and so $t = (\frac{3}{4}, \frac{1}{4}) = (.75, .25)$. Thus he predicts that, on the thousandth trial, 75% of the mice will go to the right and 25% will go to the left.

7.20. Given the transition matrix $P = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ with initial probability distribution $p^{(0)} = (\frac{1}{3}, \frac{2}{3})$. Define and find: (i) $p_{21}^{(3)}$, (ii) $p^{(3)}$, (iii) $p_2^{(3)}$.

(i) $p_{21}^{(3)}$ is the probability of moving from state a_2 to state a_1 in 3 steps. It can be obtained from the 3-step transition matrix P^3; hence first compute P^3:

$$P^2 = \begin{pmatrix} 1 & 0 \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}, \quad P^3 = \begin{pmatrix} 1 & 0 \\ \frac{7}{8} & \frac{1}{8} \end{pmatrix}$$

Then $p_{21}^{(3)}$ is the entry in the second row first column of P^3: $p_{21}^{(3)} = \frac{7}{8}$.

(ii) $p^{(3)}$ is the probability distribution of the system after three steps. It can be obtained by successively computing $p^{(1)}$, $p^{(2)}$ and then $p^{(3)}$:

$$p^{(1)} = p^{(0)}P = (\tfrac{1}{3}, \tfrac{2}{3})\begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = (\tfrac{2}{3}, \tfrac{1}{3})$$

$$p^{(2)} = p^{(1)}P = (\tfrac{2}{3}, \tfrac{1}{3})\begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = (\tfrac{5}{6}, \tfrac{1}{6})$$

$$p^{(3)} = p^{(2)}P = (\tfrac{5}{6}, \tfrac{1}{6})\begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = (\tfrac{11}{12}, \tfrac{1}{12})$$

However, since the 3-step transition matrix P^3 has already been computed in (i), $p^{(3)}$ can also be obtained as follows:

$$p^{(3)} = p^{(0)}P^3 = (\tfrac{1}{3}, \tfrac{2}{3})\begin{pmatrix} 1 & 0 \\ \frac{7}{8} & \frac{1}{8} \end{pmatrix} = (\tfrac{11}{12}, \tfrac{1}{12})$$

(iii) $p_2^{(3)}$ is the probability that the process is in the state a_2 after 3 steps; it is the second component of the 3-step probability distribution $p^{(3)}$: $p_2^{(3)} = \frac{1}{12}$.

7.21. Given the transition matrix $P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and the initial probability distribution $p^{(0)} = (\frac{2}{3}, 0, \frac{1}{3})$. Find: (i) $p_{32}^{(2)}$ and $p_{13}^{(2)}$, (ii) $p^{(4)}$ and $p_3^{(4)}$, (iii) the vector that $p^{(0)}P^n$ approaches, (iv) the matrix that P^n approaches.

(i) First compute the 2-step transition matrix P^2:

$$P^2 = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

Then $p_{32}^{(2)} = \frac{1}{2}$ and $p_{13}^{(2)} = 0$, since these numbers refer to the entries in P^2.

(ii) To compute $p^{(4)}$, use the 2-step transition matrix P^2 and the initial probability distribution $p^{(0)}$:

$$p^{(2)} = p^{(0)}P^2 = (\tfrac{1}{3}, \tfrac{2}{3}, 0) \quad \text{and} \quad p^{(4)} = p^{(2)}P^2 = (\tfrac{1}{4}, \tfrac{7}{12}, \tfrac{1}{6})$$

Since $p_3^{(4)}$ is the third component of $p^{(4)}$, $p_3^{(4)} = \frac{1}{6}$.

(iii) By Theorem 7.3, $p^{(0)}P^n$ approaches the unique fixed probability vector t of P. To obtain t, first find any fixed vector $u = (x, y, z)$:

$$(x, y, z)\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \end{pmatrix} = (x, y, z) \qquad \text{or} \qquad \begin{cases} \frac{1}{2}y = x \\ \frac{1}{2}x + \frac{1}{2}y + z = y \\ \frac{1}{2}x = z \end{cases}$$

Find any nonzero solution of the above system of equations. Set $z = 1$; then by the third equation $x = 2$, and by the first equation $y = 4$. Thus $u = (2, 4, 1)$ is a fixed point of P and so $t = (\frac{2}{7}, \frac{4}{7}, \frac{1}{7})$. In other words, $p^{(0)}P^n$ approaches $(\frac{2}{7}, \frac{4}{7}, \frac{1}{7})$.

(iv) P^n approaches the matrix T whose rows are each the fixed probability vector of P; hence

$$P^n \text{ approaches } \begin{pmatrix} \frac{2}{7} & \frac{4}{7} & \frac{1}{7} \\ \frac{2}{7} & \frac{4}{7} & \frac{1}{7} \\ \frac{2}{7} & \frac{4}{7} & \frac{1}{7} \end{pmatrix}.$$

7.22. A salesman's territory consists of three cities, A, B and C. He never sells in the same city on successive days. If he sells in city A, then the next day he sells in city B. However, if he sells in either B or C, then the next day he is twice as likely to sell in city A as in the other city. In the long run, how often does he sell in each of the cities?

The transition matrix of the problem is as follows:

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix} \end{array}$$

We seek the unique fixed probability vector t of the matrix P. First find any fixed vector $u = (x, y, z)$:

$$(x, y, z)\begin{pmatrix} 0 & 1 & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix} = (x, y, z) \qquad \text{or} \qquad \begin{cases} \frac{2}{3}y + \frac{2}{3}z = x \\ x + \frac{1}{3}z = y \\ \frac{1}{3}y = z \end{cases}$$

Set, say, $z = 1$. Then by the third equation $y = 3$, and by the first equation $x = \frac{8}{3}$. Thus $u = (\frac{8}{3}, 3, 1)$. Also $3u = (8, 9, 3)$ is a fixed vector of P. Multiply $3u$ by $1/(8 + 9 + 3) = \frac{1}{20}$ to obtain the required fixed probability vector $t = (\frac{2}{5}, \frac{9}{20}, \frac{3}{20}) = (.40, .45, .15)$. Thus in the long run he sells 40% of the time in city A, 45% of the time in B and 15% of the time in C.

7.23. There are 2 white marbles in urn A and 3 red marbles in urn B. At each step of the process a marble is selected from each urn and the two marbles selected are interchanged. Let the state a_i of the system be the number i of red marbles in urn A. (i) Find the transition matrix P. (ii) What is the probability that there are 2 red marbles in urn A after 3 steps? (iii) In the long run, what is the probability that there are 2 red marbles in urn A?

(i) There are three states, a_0, a_1 and a_2 described by the following diagrams:

2 W	3 R	1 W, 1 R	1 W, 2 R	2 R	2 W, 1 R
A	B	A	B	A	B
a_0		a_1		a_2	

If the system is in state a_0, then a white marble must be selected from urn A and a red marble from urn B, so the system must move to state a_1. Accordingly, the first row of the transition matrix is $(0, 1, 0)$.

Suppose the system is in state a_1. It can move to state a_0 if and only if a red marble is selected from urn A and a white marble from urn B; the probability of that happening is $\frac{1}{2}\cdot\frac{1}{3}=\frac{1}{6}$. Thus $p_{10}=\frac{1}{6}$. The system can move from state a_1 to a_2 if and only if a white marble is selected from urn A and a red marble from urn B; the probability of that happening is $\frac{1}{2}\cdot\frac{2}{3}=\frac{1}{3}$. Thus $p_{12}=\frac{1}{3}$. Accordingly, the probability that the system remains in state a_1 is $p_{11}=1-\frac{1}{6}-\frac{1}{3}=\frac{1}{2}$. Thus the second row of the transition matrix is $(\frac{1}{6},\frac{1}{2},\frac{1}{3})$. (Note that p_{11} can also be obtained from the fact that the system remains in the state a_1 if either a white marble is drawn from each urn, probability $\frac{1}{2}\cdot\frac{1}{3}=\frac{1}{6}$, or a red marble is drawn from each urn, probability $\frac{1}{2}\cdot\frac{2}{3}=\frac{1}{3}$; thus $p_{11}=\frac{1}{6}+\frac{1}{3}=\frac{1}{2}$.)

Now suppose the system is in state a_2. A red marble must be drawn from urn A. If a red marble is selected from urn B, probability $\frac{1}{3}$, then the system remains in state a_2; and if a white marble is selected from urn B, probability $\frac{2}{3}$, then the system moves to state a_1. Note that the system can never move from state a_2 to the state a_0. Thus the third row of the transition matrix is $(0,\frac{2}{3},\frac{1}{3})$. That is,

$$P \;=\; \begin{array}{c} \\ a_0 \\ a_1 \\ a_2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} a_0 & a_1 & a_2 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix} \end{array}$$

(ii) The system began in state a_0, i.e. $p^{(0)}=(1,0,0)$. Thus:

$$p^{(1)} = p^{(0)}P = (0,1,0), \quad p^{(2)} = p^{(1)}P = (\tfrac{1}{6},\tfrac{1}{2},\tfrac{1}{3}), \quad p^{(3)} = p^{(2)}P = (\tfrac{1}{12},\tfrac{23}{36},\tfrac{5}{18})$$

Accordingly, the probability that there are 2 red marbles in urn A after 3 steps is $\frac{5}{18}$.

(iii) We seek the unique fixed probability vector t of the transition matrix P. First find any fixed vector $u=(x,y,z)$:

$$(x,y,z)\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix} = (x,y,z) \qquad \text{or} \qquad \begin{cases} \frac{1}{6}y = x \\ x+\frac{1}{2}y+\frac{2}{3}z = y \\ \frac{1}{3}y+\frac{1}{3}z = z \end{cases}$$

Set, say, $x=1$. Then by the first equation $y=6$, and by the third equation $z=3$. Hence $u=(1,6,3)$. Multiply u by $1/(1+6+3)=\frac{1}{10}$ to obtain the required unique fixed probability vector $t=(.1,.6,.3)$. Thus, in the long run, 30% of the time there will be 2 red marbles in urn A.

Note that the long run probability distribution is the same as if the five marbles were placed in an urn and 2 were selected at random to put into urn A.

7.24. A player has \$2. He bets \$1 at a time and wins \$1 with probability $\frac{1}{2}$. He stops playing if he loses the \$2 or wins \$4. (i) What is the probability that he has lost his money at the end of, at most, 5 plays? (ii) What is the probability that the game lasts more than 7 plays?

This is a random walk with absorbing barriers at 0 and 6 (see Examples 7.22 and 7.23). The transition matrix is

$$P \;=\; \begin{array}{c} \\ a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{array}\!\!\begin{array}{c} \begin{array}{ccccccc} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

with initial probability distribution $p^{(0)}=(0,0,1,0,0,0,0)$ since he began with \$2.

(i) We seek $p_0^{(5)}$, the probability that the system is in state a_0 after five steps. Compute the 5th step probability distribution $p^{(5)}$:

$$p^{(1)} = p^{(0)}P = (0, \tfrac{1}{2}, 0, \tfrac{1}{2}, 0, 0, 0) \qquad p^{(4)} = p^{(3)}P = (\tfrac{3}{8}, 0, \tfrac{5}{16}, 0, \tfrac{1}{4}, 0, \tfrac{1}{16})$$
$$p^{(2)} = p^{(1)}P = (\tfrac{1}{4}, 0, \tfrac{1}{2}, 0, \tfrac{1}{4}, 0, 0) \qquad p^{(5)} = p^{(4)}P = (\tfrac{3}{8}, \tfrac{5}{32}, 0, \tfrac{9}{32}, 0, \tfrac{1}{8}, \tfrac{1}{16})$$
$$p^{(3)} = p^{(2)}P = (\tfrac{1}{4}, \tfrac{1}{4}, 0, \tfrac{3}{8}, 0, \tfrac{1}{8}, 0)$$

Thus $p_0^{(5)}$, the probability that he has no money after 5 plays, is $\frac{3}{8}$.

(ii) Compute $p^{(7)}$: $p^{(6)} = p^{(5)}P = (\frac{29}{64}, 0, \frac{7}{32}, 0, \frac{13}{64}, 0, \frac{1}{8})$. $p^{(7)} = p^{(6)}P = (\frac{29}{64}, \frac{7}{64}, 0, \frac{27}{128}, 0, \frac{13}{128}, \frac{1}{8})$

The probability that the game lasts more than 7 plays, i.e. that the system is not in state a_0 or a_6 after 7 steps, is $\frac{7}{64} + \frac{27}{128} + \frac{13}{128} = \frac{27}{64}$.

7.25. Consider repeated tosses of a fair die. Let X_n be the maximum of the numbers occurring in the first n trials.

(i) Find the transition matrix P of the Markov chain. Is the matrix regular?

(ii) Find $p^{(1)}$, the probability distribution after the first toss.

(iii) Find $p^{(2)}$ and $p^{(3)}$.

(i) The state space of the Markov chain is $\{1, 2, 3, 4, 5, 6\}$. The transition matrix is

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{2}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & \frac{3}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & \frac{4}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

We obtain, for example, the third row of the matrix as follows. Suppose the system is in state 3, i.e. the maximum of the numbers occurring on the first n trials is 3. Then the system remains in state 3 if a 1, 2, or 3 occurs on the $(n+1)$-st trial; hence $p_{33} = \frac{3}{6}$. On the other hand, the system moves to state 4, 5 or 6, respectively, if a 4, 5 or 6 occurs on the $(n+1)$-st trial; hence $p_{34} = p_{35} = p_{36} = \frac{1}{6}$. The system can never move to state 1 or 2 since a 3 has occurred on one of the trials; hence $p_{31} = p_{32} = 0$. Thus the third row of the transition matrix is $(0, 0, \frac{3}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$. The other rows are obtained similarly.

The matrix is not regular since state 6 is absorbing, i.e. there is a 1 on the main diagonal in row 6.

(ii) On the first toss of the die, the state of the system X_1 is the number occurring; hence $p^{(1)} = (\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$.

(iii) $p^{(2)} = p^{(1)}P = (\frac{1}{36}, \frac{3}{36}, \frac{5}{36}, \frac{7}{36}, \frac{9}{36}, \frac{11}{36})$. $p^{(3)} = p^{(2)}P = (\frac{1}{216}, \frac{7}{216}, \frac{19}{216}, \frac{37}{216}, \frac{61}{216}, \frac{91}{216})$.

7.26. Two boys b_1 and b_2 and two girls g_1 and g_2 are throwing a ball from one to the other. Each boy throws the ball to the other boy with probability $\frac{1}{2}$ and to each girl with probability $\frac{1}{4}$. On the other hand, each girl throws the ball to each boy with probability $\frac{1}{2}$ and never to the other girl. In the long run, how often does each receive the ball?

This is a Markov chain with state space $\{b_1, b_2, g_1, g_2\}$ and transition matrix

$$P = \begin{array}{c} \\ b_1 \\ b_2 \\ g_1 \\ g_2 \end{array} \begin{array}{c} \begin{array}{cccc} b_1 & b_2 & g_1 & g_2 \end{array} \\ \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix} \end{array}$$

We seek a fixed vector $u = (x, y, z, w)$ of P: $(x, y, z, w)P = (x, y, z, w)$. Set the corresponding components of uP equal to u to obtain the system

$$\begin{aligned} \tfrac{1}{2}y + \tfrac{1}{2}z + \tfrac{1}{2}w &= x \\ \tfrac{1}{2}x + \tfrac{1}{2}z + \tfrac{1}{2}w &= y \\ \tfrac{1}{4}x + \tfrac{1}{4}y &= z \\ \tfrac{1}{4}x + \tfrac{1}{4}y &= w \end{aligned}$$

We seek any nonzero solution. Set, say, $z = 1$; then $w = 1$, $x = 2$ and $y = 2$. Thus $u = (2, 2, 1, 1)$ and so the unique fixed probability of P is $t = (\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6})$. Thus, in the long run, each boy receives the ball $\frac{1}{3}$ of the time and each girl $\frac{1}{6}$ of the time.

7.27. Prove Theorem 7.6: Let $P = (p_{ij})$ be the transition matrix of a Markov chain. If $p = (p_i)$ is the probability distribution of the system at some arbitrary time k, then pP is the probability distribution of the system one step later, i.e. at time $k + 1$; hence pP^n is the probability distribution of the system n steps later, i.e. at time $k + n$. In particular, $p^{(1)} = p^{(0)}P$, $p^{(2)} = p^{(1)}P$, ... and also $p^{(n)} = p^{(0)}P^n$.

Suppose the state space is $\{a_1, a_2, \ldots, a_m\}$. The probability that the system is in state a_j at time k and then in state a_i at time $k + 1$ is the product $p_j p_{ji}$. Thus the probability that the system is in state a_i at time $k + 1$ is the sum

$$p_1 p_{1i} + p_2 p_{2i} + \cdots + p_m p_{mi} = \sum_{j=1}^{m} p_j p_{ji}$$

Thus the probability distribution at time $k + 1$ is

$$p^* = \left(\sum_{j=1}^{m} p_j p_{j1}, \sum_{j=1}^{m} p_j p_{j2}, \ldots, \sum_{j=1}^{m} p_j p_{jm} \right)$$

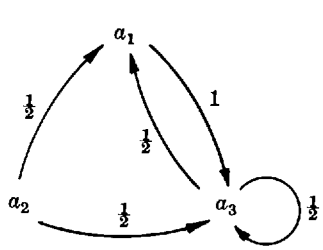
However, this vector is precisely the product of the vector $p = (p_i)$ by the matrix $P = (p_{ij})$: $p^* = pP$.

7.28. Prove Theorem 7.5: Let P be the transition matrix of a Markov chain. Then the n-step transition matrix is equal to the nth power of P: $P^{(n)} = P^n$.

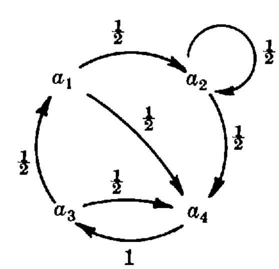
Suppose the system is in state a_i at, say, time k. We seek the probability $p_{ij}^{(n)}$ that the system is in state a_j at time $k + n$. Now the probability distribution of the system at time k, since the system is in state a_i, is the vector $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ which has a 1 at the ith position and zeros everywhere else. By the preceding problem, the probability distribution at time $k + n$ is the product $e_i P^n$. But $e_i P^n$ is the ith row of the matrix P^n. Thus $P_{ij}^{(n)}$ is the jth component of the ith row of P^n, and so $P^{(n)} = P^n$.

MISCELLANEOUS PROBLEMS

7.29. The transition probabilities of a Markov chain can be represented by a diagram, called a *transition diagram*, where a positive probability p_{ij} is denoted by an arrow from the state a_i to the state a_j. Find the transition matrix of each of the following transition diagrams:



(i)



(ii)

(i) Note first that the state space is $\{a_1, a_2, a_3\}$ and so the transition matrix is of the form

$$P \;=\; \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \left(\begin{array}{ccc} & & \\ & & \\ & & \end{array}\right) \end{array}$$

The ith row of the matrix is obtained by finding those arrows which emanate from a_i in the diagram; the number attached to the arrow from a_i to a_j is the jth component of the ith row. Thus the transition matrix is

$$P \;=\; \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \left(\begin{array}{ccc} 0 & 0 & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{array}\right) \end{array}$$

(ii) The state space is $\{a_1, a_2, a_3, a_4\}$. The transition matrix is

$$P \;=\; \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \left(\begin{array}{cccc} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \end{array}\right) \end{array}$$

7.30. Suppose the transition matrix of a Markov chain is as follows:

$$P \;=\; \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \left(\begin{array}{cccc} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{array}\right) \end{array}$$

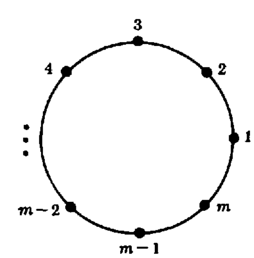
Is the Markov chain regular?

Note that once the system enters the state a_1 or the state a_2, then it can never move to state a_3 or state a_4, i.e. the system remains in the state subspace $\{a_1, a_2\}$. Thus, in particular, $p_{13}^{(n)} = 0$ for every n and so every power P^n will contain a zero entry. Hence P is not regular.

7.31. Suppose m points on a circle are numbered respectively $1, 2, \ldots, m$ in a counterclockwise direction. A particle performs a "random walk" on the circle; it moves one step counterclockwise with probability p or one step clockwise with probability $q = 1 - p$. Find the transition matrix of this Markov chain.

The state space is $\{1, 2, \ldots, m\}$. The diagram to the right below can be used to obtain the transition matrix which appears to the left below.

$$P \;=\; \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ m-1 \\ m \end{array}\!\!\begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & \cdots & m-2 & m-1 & m \end{array} \\ \left(\begin{array}{cccccccc} 0 & p & 0 & 0 & \cdots & 0 & 0 & q \\ q & 0 & p & 0 & \cdots & 0 & 0 & 0 \\ 0 & q & 0 & p & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & q & 0 & p \\ p & 0 & 0 & 0 & \cdots & 0 & q & 0 \end{array}\right) \end{array}$$



Supplementary Problems

MATRIX MULTIPLICATION

7.32. Given $A = \begin{pmatrix} 1 & -2 & 3 \\ 4 & 1 & -1 \\ 5 & 2 & 3 \end{pmatrix}$. Find uA if (i) $u = (1, -3, 2)$, (ii) $u = (3, 0, -2)$, (iii) $u = (4, -1, -1)$.

7.33. Given $A = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 1 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ 6 & -3 \\ 1 & -2 \end{pmatrix}$. Find AB and BA.

7.34. Given $A = \begin{pmatrix} 2 & 2 \\ 3 & -1 \end{pmatrix}$. Find A^2 and A^3.

7.35. Given $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. Find A^n.

PROBABILITY VECTORS AND STOCHASTIC MATRICES

7.36. Which vectors are probability vectors?

(i) $(\frac{1}{4}, \frac{1}{2}, -\frac{1}{4}, \frac{1}{2})$ (ii) $(\frac{1}{2}, 0, \frac{1}{3}, \frac{1}{6}, \frac{1}{6})$ (iii) $(\frac{1}{12}, \frac{1}{2}, \frac{1}{6}, 0, \frac{1}{4})$.

7.37. Find a scalar multiple of each vector which is a probability vector:

(i) $(3, 0, 2, 5, 3)$ (ii) $(2, \frac{1}{2}, 0, \frac{1}{4}, \frac{3}{4}, 0, 1)$ (iii) $(\frac{1}{3}, 2, \frac{1}{2}, 0, \frac{1}{4}, \frac{2}{3})$.

7.38. Which matrices are stochastic?

(i) $\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (iii) $\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{4} \end{pmatrix}$ (iv) $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ (v) $\begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}$

REGULAR STOCHASTIC MATRICES AND FIXED PROBABILITY VECTORS

7.39. Find the unique fixed probability vector of each matrix:

(i) $\begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{5} & \frac{3}{5} \end{pmatrix}$ (ii) $\begin{pmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{5}{6} & \frac{1}{6} \end{pmatrix}$ (iii) $\begin{pmatrix} .2 & .8 \\ .5 & .5 \end{pmatrix}$ (iv) $\begin{pmatrix} .7 & .3 \\ .6 & .4 \end{pmatrix}$

7.40. (i) Find the unique fixed probability vector t of $P = \begin{pmatrix} 0 & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

(ii) What matrix does P^n approach? (iii) What vector does $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})P^n$ approach?

7.41. Find the unique fixed probability vector t of each matrix:

(i) $A = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 1 & 0 \end{pmatrix}$ (ii) $B = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$

7.42. (i) Find the unique fixed probability vector t of $P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$.

(ii) What matrix does P^n approach?

(iii) What vector does $(\frac{1}{4}, 0, \frac{1}{2}, \frac{1}{4})P^n$ approach?

(iv) What vector does $(\frac{1}{2}, 0, 0, \frac{1}{2})P^n$ approach?

7.43. (i) Given that $t = (\frac{1}{2}, 0, \frac{1}{4}, \frac{1}{4})$ is a fixed point of a stochastic matrix P, is P regular?
(ii) Given that $t = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ is a fixed point of a stochastic matrix P, is P regular?

7.44. Which of the stochastic matrices are regular?

$$\text{(i)} \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \quad \text{(ii)} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \quad \text{(iii)} \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix}$$

7.45. Show that $(cf + ce + de,\ af + bf + ae,\ ad + bd + bc)$ is a fixed point of the matrix

$$P = \begin{pmatrix} 1-a-b & a & b \\ c & 1-c-d & d \\ e & f & 1-e-f \end{pmatrix}$$

MARKOV CHAINS

7.46. A man's smoking habits are as follows. If he smokes filter cigarettes one week, he switches to nonfilter cigarettes the next week with probability .2. On the other hand, if he smokes nonfilter cigarettes one week, there is a probability of .7 that he will smoke nonfilter cigarettes the next week as well. In the long run, how often does he smoke filter cigarettes?

7.47. A gambler's luck follows a pattern. If he wins a game, the probability of winning the next game is .6. However, if he loses a game, the probability of losing the next game is .7. There is an even chance that the gambler wins the first game.
(i) What is the probability that he wins the second game?
(ii) What is the probability that he wins the third game?
(iii) In the long run, how often will he win?

7.48. For a Markov chain, the transition matrix is $P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$ with initial probability distribution $p^{(0)} = (\frac{1}{4}, \frac{3}{4})$. Find: (i) $p_{21}^{(2)}$; (ii) $p_{12}^{(2)}$; (iii) $p^{(2)}$; (iv) $p_1^{(2)}$; (v) the vector $p^{(0)}P^n$ approaches; (vi) the matrix P^n approaches.

7.49. For a Markov chain, the transition matrix is $P = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}$ and the initial probability distribution is $p^{(0)} = (\frac{1}{2}, \frac{1}{2}, 0)$. Find (i) $p_{13}^{(2)}$, (ii) $p_{23}^{(2)}$, (iii) $p^{(2)}$, (iv) $p_1^{(2)}$.

7.50. Each year a man trades his car for a new car. If he has a Buick, he trades it for a Plymouth. If he has a Plymouth, he trades it for a Ford. However, if he has a Ford, he is just as likely to trade it for a new Ford as to trade it for a Buick or a Plymouth. In 1955 he bought his first car which was a Ford.
(i) Find the probability that he has a (*a*) 1957 Ford, (*b*) 1957 Buick, (*c*) 1958 Plymouth, (*d*) 1958 Ford.
(ii) In the long run, how often will he have a Ford?

7.51. There are 2 white marbles in urn A and 4 red marbles in urn B. At each step of the process a marble is selected from each urn, and the two marbles selected are interchanged. Let X_n be the number of red marbles in urn A after n interchanges. (i) Find the transition matrix P. (ii) What is the probability that there are 2 red marbles in urn A after 3 steps? (iii) In the long run, what is the probability that there are 2 red marbles in urn A?

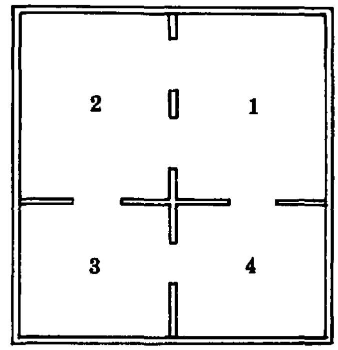
7.52. Solve the preceding problem in the case that there are 3 white marbles in urn A and 3 red marbles in urn B.

7.53. A fair coin is tossed until 3 heads occur in a row. Let X_n be the length of the sequence of heads ending at the nth trial. (See Example 7.24.) What is the probability that there are at least 8 tosses of the coin?

7.54. A player has 3 dollars. At each play of a game, he loses one dollar with probability $\frac{3}{4}$ but wins two dollars with probability $\frac{1}{4}$. He stops playing if he has lost his 3 dollars or he has won at least 3 dollars.

(i) Find the transition matrix of the Markov chain.

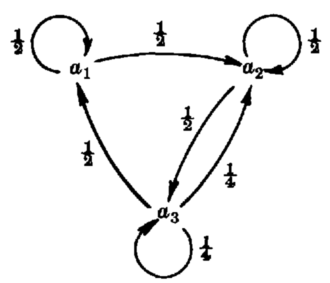
(ii) What is the probability that there are at least 4 plays to the game?



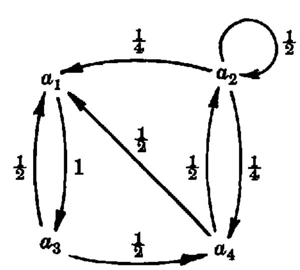
7.55. The diagram on the right shows four compartments with doors leading from one to another. A mouse in any compartment is equally likely to pass through each of the doors of the compartment. Find the transition matrix of the Markov chain.

MISCELLANEOUS PROBLEMS

7.56. Find the transition matrix corresponding to each transition diagram:



(i)



(ii)

7.57. Draw a transition diagram for each transition matrix:

(i) $P = \begin{array}{c c} & \begin{array}{cc} a_1 & a_2 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \end{array} & \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \end{array}$ (ii) $P = \begin{array}{c c} & \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} & \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \end{array}$

7.58. Consider the vector $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ which has a 1 at the ith position and zeros elsewhere. Show that e_iA is the ith row of the matrix A (whenever the product is defined).

Answers to Supplementary Problems

7.32. (i) $(-1, -1, 12)$, (ii) $(-7, -10, 3)$, (iii) $(-5, -11, 10)$

7.33. $AB = \begin{pmatrix} 0 & -4 \\ 17 & -10 \end{pmatrix}$, $BA = \begin{pmatrix} 5 & -1 & 13 \\ -3 & -9 & 9 \\ -5 & -3 & -6 \end{pmatrix}$

7.34. $A^2 = \begin{pmatrix} 10 & 2 \\ 3 & 7 \end{pmatrix}$, $A^3 = \begin{pmatrix} 26 & 18 \\ 27 & -1 \end{pmatrix}$

7.35. $A^n = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}$

7.36. Only (iii).

7.37. (i) (3/13, 0, 2/13, 5/13, 3/13)
(ii) (8/18, 2/18, 0, 1/18, 3/18, 0, 4/18)
(iii) (4/45, 24/45, 6/45, 0, 3/45, 8/45)

7.38. Only (ii) and (iv).

7.39. (i) (6/11, 5/11), (ii) (10/19, 9/19), (iii) (5/13, 8/13), (iv) $(\frac{2}{3}, \frac{1}{3})$

7.40. (i) t = (4/13, 8/13, 1/13), (iii) t = (4/13, 8/13, 1/13)

7.41. (i) t = (2/9, 6/9, 1/9), (ii) t = (5/15, 6/15, 4/15)

7.42. (i) t = (2/11, 4/11, 1/11, 4/11), (iii) t, (iv) t

7.43. (i) No, (ii) not necessarily, e.g. $P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

7.44. Only (iii)

7.46. 60% of the time

7.47. (i) 9/20, (ii) 87/200, (iii) 3/7 of the time

7.48. (i) 9/16, (ii) 3/8, (iii) (37/64, 27/64), (iv) 37/64, (v) (.6, .4), (vi) $\begin{pmatrix} .6 & .4 \\ .6 & .4 \end{pmatrix}$

7.49. (i) 3/8, (ii) 1/2, (iii) (7/16, 2/16, 7/16), (iv) 7/16

7.50. (i) (*a*) 4/9, (*b*) 1/9, (*c*) 7/27, (*d*) 16/27. (ii) 50% of the time

7.51. (i) $P = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{8} & \frac{1}{2} & \frac{3}{8} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ (ii) 3/8 (iii) 2/5

7.52. (i) $P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{9} & \frac{4}{9} & \frac{4}{9} & 0 \\ 0 & \frac{4}{9} & \frac{4}{9} & \frac{1}{9} \\ 0 & 0 & 1 & 0 \end{pmatrix}$ (ii) 32/81 (iii) 9/20

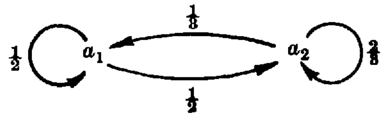
7.53. 81/128

7.54. (i) $P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{3}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{3}{4} & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{3}{4} & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ (ii) 27/64

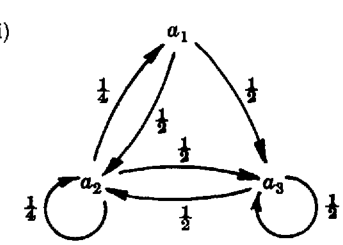
7.55. $P = \begin{pmatrix} 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}$

7.56. (i) $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$ (ii) $\begin{pmatrix} 0 & 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix}$

7.57. (i)



(ii)



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